

# GROUP EXTENSIONS WITH INFINITE CONJUGACY CLASSES

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**ABSTRACT.** We characterize the group property of being with infinite conjugacy classes (or *icc*, *i.e.* infinite and of which all conjugacy classes except  $\{1\}$  are infinite) for groups which are defined by an extension of groups. We give characterizations for all different kinds of extension: direct product, semi-direct product, wreath products and general extension. We also give many particular results when the groups involved verify some additional hypothesis.

The *icc* property is correlated to the Theory of Von Neumann algebras since a necessary and sufficient condition for the Von Neumann algebra of a group  $\Gamma$  to be a factor of type  $II - 1$ , is that  $\Gamma$  be *icc*. Our approach applies in full generality in the study of *icc* property since any group either decomposes as an extension of groups or is simple, and in the latter case *icc* property becomes trivially equivalent to being infinite.

## INTRODUCTION

A group is said to be with *infinite conjugacy classes* (or *icc*) if it is infinite, and if all its conjugacy classes except  $\{1\}$  are infinite. For example non-abelian free groups and more generally non-elementary torsion free hyperbolic groups, are *icc*, while abelian or finite groups are not. This property has been motivated by the theory of operator algebras, specifically by Von Neumann algebras through the Murray-Von Neumann characterization of factor of type  $II - 1$  (cf. [ROIV]):

**Murray-Von Neumann.** *A necessary and sufficient condition for the Von Neumann algebra of a group  $\Gamma$  to be a factor of type  $II - 1$  is that  $\Gamma$  be *icc*.*

The property of being *icc* has been characterized in several classes of groups : 3-manifolds and  $PD(3)$  groups in [HP], amalgamated products and HNN-extensions in [Co]. We will focus here on groups defined by an extension of groups *i.e.* on non-simple groups, and will characterize the property by mean of the invariants associated to the extension. Recall that a group  $G$  is said to decompose as an extension whenever  $G$  contains a proper non-trivial normal subgroup  $K$ , which gives rise to a short exact sequence of groups:

$$1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1 .$$

Towards this direction particular results are already known. In [HP] (lemmas 5 and 6) are proved the following results:

- Let  $G$  be defined by a short exact sequence :

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

Then  $G$  is *icc* if and only if  $K$  is *icc* and  $G \not\cong K \times \mathbb{Z}/2\mathbb{Z}$ . (It is easily seen that the condition can be rephrased as:  $K$  *icc* and the associated homomorphism  $\mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Out}(K)$  is injective.)

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• Let  $G$  be a torsion-free group containing a finite index subgroup  $K$ . Then  $G$  is icc if and only if  $K$  is icc.

In [Co], as a particular case of HNN extensions, has been proved the following result on extensions by  $\mathbb{Z}$ :

• Let  $G = D \rtimes \mathbb{Z}$  with  $D \neq \{1\}$  and  $\pi : G \rightarrow \text{Aut}(D)$  the associated map; then  $G$  is not icc if and only if one of the following conditions is satisfied:

- (i)  $D$  contains a normal subgroup  $N \neq \{1\}$  preserved by  $\pi(G)$  and either  $N$  is finite or  $N = \mathbb{Z}^n$  and the natural homomorphism from  $\pi(G)$  to  $GL(n, \mathbb{Z})$  has a finite image,
- (ii) the associated map  $\theta : \mathbb{Z} \rightarrow \text{Out}(D)$  is non injective.

We treat separately all different kinds of extensions (semi-direct products, wreath products, general extensions), and characterize the property by mean of the values associated to the extension. We give numerous particularizations when the groups involved (kernel or quotient in the extension) verify some additional hypothesis and the result becomes more concise. We also apply the results in case of amalgams and HNN extensions of groups, as well as finite index subgroups.

After a first section dedicated to preliminaries we treat in section 2 the case of wreath products of groups, both complete and restricted. In section 3 we treat the case of semi-direct products, and in section 4 the case of extensions of groups in full generality. Section 5 is devoted to particular cases of extensions for which the result can be improved, and in section 6 as examples we give results concerning the icc property for amalgams and HNN-extensions (shortly recovering results proved in [Co]), and groups containing a finite index subgroup.

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## 1. PRELIMINARIES

We first fix some notations: let  $G$  be a group and let  $x, y \in G$ , then  ${}^y x$  is the element of  $G$  defined by  ${}^y x := yxy^{-1}$ . If  $H$  is a subgroup of  $G$ , then  ${}^H x = \{{}^y x ; y \in H\}$ ; in particular  ${}^G x$  denote the conjugacy class of  $x$  in  $G$ . The centralizer of  $x$  in  $H$  is  $Z_H(x) := \{y \in H ; [x, y] = 1\}$ , (where  $[x, y] := xyx^{-1}y^{-1}$ ), and the center of  $G$  is  $Z(G) := \bigcap_{x \in G} Z_G(x)$ .

The following properties are straightforward:

**Property 1.** *The conjugacy class  ${}^G x$  of  $x$  in  $G$  is finite if and only if the centralizer  $Z_G(x)$  of  $x$  has a finite index in  $G$ . More precisely the cardinal of  ${}^G x$  equals  $[G : Z_G(x)]$ .*

In particular, the center  $Z(G)$  of a group  $G$  consists of those elements whose conjugacy class is reduced to a single element, and one obtains trivially:

**Property 2.** *If  $G$  has a non-trivial center,  $Z(G) \neq \{1\}$ , then  $G$  is not icc.*

*Example.* Nilpotent groups are not icc, for a non-trivial nilpotent group has a non-trivial center (cf. Theorem 5.34, [Ro]).

The reciprocal of property 2 is false; one may consider for example the infinite dihedral group  $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$  which has trivial center while it is not icc since it contains an infinite cyclic subgroup with index 2.

By applying property 2 one obtains the straightforward:

**Theorem 1.1** (central extensions are not icc). *A group which decomposes as a central extension is not icc.*

**Property 3.** *Let  $f : G \rightarrow H$  be an homomorphism of groups and  $x \in G$ ,*  
*– if  $f$  surjects and  ${}^Gx$  is finite then  ${}^Hf(x)$  is finite,*  
*– if  $f$  injects and  ${}^Gx$  is infinite then  ${}^Hf(x)$  is infinite.*

As an immediate application of property 3 one obtains the obvious:

**Theorem 1.2** (direct product icc). *Let  $A, B$  be non-trivial groups. The direct product  $A \times B$  is icc if and only if  $A$  and  $B$  are icc.*

An *FC*-group is a group whom all conjugacy classes are finite. Such groups have been extensively studied and are almost well understood.

**Property 4.** (cf. [Ne]) *In an FC-group  $G$  the torsion elements form a normal subgroup  $Tor(G)$  and  $G/Tor(G)$  is a torsion-free abelian group. If  $G$  is finitely generated both  $Tor(G)$  and  $G/Z(G)$  are finite.*

As a consequence we obtain:

**Property 5** (simple group icc). *A simple group is icc if and only if it is infinite.*

*Proof.* Obviously an icc group must be infinite. Let  $G \neq \{1\}$  be a simple group. Suppose  $G$  is not icc; let  $\omega \neq 1$  with  ${}^G\omega$  finite;  ${}^G\omega$  generates a normal subgroup in  $G$ . Hence  $G$  is a finitely generated *FC*-group. Now  $Tor(G) = G$ , since torsion-free abelian groups are not simple, and  $G$  is finite (property 4).  $\square$

If  $G$  is not simple then it decomposes as an extension of groups. Hence while for simple groups being icc is a trivial property equivalent to being infinite, in the study of icc property the case of extensions of groups appears as the more delicate one to deal with.

We will make a heavy use of the following straightforward fact together with property 4.

**Property 6.** *The union of all finite conjugacy classes in a group  $G$  is a characteristic subgroup of  $G$  that we shall denote  $FC(G)$ . In particular  $FC(G)$  is an FC-group which is normal in  $G$ .*

Note that obviously  $G$  is icc if and only if  $FC(G) = \{1\}$  and that  $FC(G)$  contains  $Z(G)$  as a normal subgroup.

## 2. WREATH PRODUCTS

Throughout the section,  $D, Q$  are groups and  $\Omega$  is a  $Q$ -set, *i.e.* a set equipped with a left  $Q$ -action. Let  $G$  be the *complete* (or *unrestricted*) *wreath product* denoted by  $G = D \wr_{\Omega} Q$ . That is, let us denote by  $D^{\Omega}$  the group of maps from  $\Omega$  to  $D$  and by  $\lambda : Q \rightarrow Aut(D^{\Omega})$  the homomorphism defined by  $\forall x \in \Omega, \forall \phi \in D^{\Omega}, \lambda(q)(\phi)(x) = \phi(q^{-1}x)$ ; the group  $G$  is defined to be the split extension  $G = D^{\Omega} \rtimes Q$  associated with  $\lambda$ , in the sense that  $\forall \phi \in D^{\Omega}, \forall q \in Q, q\phi q^{-1} = \lambda(q)(\phi)$ . When  $\Omega = Q$  with  $Q$  acting by left multiplication one talks of the *complete regular wreath product* denoted by  $D \wr Q$ .

We also consider the *restricted wreath product*  $G = D \wr_{\Omega, r} Q$ : let  $D^{(\Omega)}$  be the group of maps from  $\Omega$  to  $D$  with finite support, and define as above  $G$  as the split extension  $G = D^{(\Omega)} \rtimes Q$  associated with  $\lambda$ . It's a subgroup of  $D \wr_{\Omega} Q$  which is countable whenever  $D$  and  $Q$  are countable. When  $\Omega = Q$ , one talks of the *restricted regular wreath product*  $D \wr_r Q$ .

**Theorem 2.1** (wreath products icc). *Let  $G = D \wr_{\Omega} Q$  (respectively  $G = D \wr_{\Omega, r} Q$ ), with  $D \neq \{1\}$  ; a necessary and sufficient condition for  $G$  to be icc is that on the one hand condition (i) is satisfied:*

- (i) *1 is the only element of  $FC(Q)$  which fixes  $\Omega$  pointwise,*
- and on the other at least one of the following conditions is satisfied :*
- (ii)  *$D$  is icc,*
- (iii) *all  $Q$ -orbits in  $\Omega$  are infinite.*

We immediately obtain for regular wreath products:

**Corollary 2.1** (regular wreath products icc). *If the action of  $Q$  on  $\Omega$  is free (in particular for  $D \wr Q$  and  $D \wr_r Q$ ) then  $G$  is icc if and only if either  $D$  is icc or  $Q$  is infinite.*

*Example.* The lamplighter group  $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$  is icc as any regular wreath products  $D \wr \mathbb{Z}$  and  $D \wr_r \mathbb{Z}$  for  $D \neq \{1\}$ . Let  $\Omega = \mathbb{Z}/n\mathbb{Z}$  be equipped with the natural  $\mathbb{Z}$ -action ( $p, q \bmod n \mapsto p + q \bmod n$ ), then  $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z} \wr_{\Omega, r} \mathbb{Z}$  are not icc, indeed they are FC-groups (conditions (ii) and (iii) fail).

*Proof of theorem 2.1.* We treat the case of restricted wreath products; the arguments remain valid for complete wreath products by changing  $D^{(\Omega)}$  into  $D^{\Omega}$ .

In the following  $\varepsilon$  will denote the neutral element of  $D^{(\Omega)}$ , i.e. the element of  $D^{(\Omega)}$  defined by  $\forall x \in \Omega, \varepsilon(x) = 1$ . Given  $y \in \Omega$  and  $d \in D$ , the element  $\eta_d^y$  of  $D^{(\Omega)}$  is defined by  $\eta_d^y(x) = d$  if  $x = y$  and  $\eta_d^y(x) = 1$  otherwise. Elements in  $G$  will be given as couples  $(d, q) \in D^{(\Omega)} \times Q$  together with the product:  $(d_1, q_1)(d_2, q_2) = (d_1^{q_1} d_2, q_1 q_2)$ .

$G$  icc  $\implies$  condition (i) and either condition (ii) or (iii).

We first suppose that  $G$  is icc and prove the necessary part of the assumption. Necessarily condition (i) is satisfied ; otherwise there would exist  $q_0 \neq 1$  in  $FC(Q)$  fixing  $\Omega$  pointwise, and  $\{(\varepsilon, q_0) \in G \mid q \in Q\}$  would be a finite subset of  $G$  invariant under conjugacy, contradicting that  $G$  is icc. We now prove that if condition (iii) does not hold then condition (ii) must hold. Suppose that  $\Omega$  has a finite  $Q$ -orbit  $\mathcal{O}$ . If  $D$  would contain a finite conjugacy class  $\xi \neq \{1\}$ , then the set  $\Phi$  of maps from  $\Omega$  to  $\xi$  having their support in  $\mathcal{O}$  would be finite and non empty, and the subset  $\{(\phi, 1) \in G \mid \phi \in \Phi\}$  of  $G$  would be finite and invariant under conjugacy, which is impossible. Hence if  $G$  is icc, either condition (ii) or condition (iii) is satisfied, which proves the necessary part of the assumption.

We now prove the sufficient part of the assumption.

Conditions (i) and (iii)  $\implies G$  is icc.

Suppose that conditions (i) and (iii) are both satisfied. Let  $g = (\phi, q) \in G$  ; suppose first that  $\phi \neq \varepsilon$ . Its support is non empty and has an infinite  $Q$ -orbit so that  $\phi \in D^{(\Omega)}$  has an infinity of translated under the action of  $\lambda(Q)$ , and it follows that  $Qg$  and hence also  ${}^Gg$  is infinite. Suppose now that  $g = (\varepsilon, q)$  is non-trivial in  $G$ . If  $q \notin FC(Q)$  then  $Qg$  and hence also  ${}^Gg$  is infinite. If  $q \in FC(Q)$  let  $y \in \Omega$  be an element that  $q$  does not fix (existence follows from condition (i)), and  $d \neq 1$  be an element of  $D$ . Consider the element  $g' = (\eta_d^y, 1)^{-1} g (\eta_d^y, 1) = (\phi, q)$  of  $G$  ;  $g'$  is a conjugate of  $g$  and  $\phi = \eta_{d^{-1}}^y \eta_d^{qy} \neq \varepsilon$ . Hence the above argument applies to show that  ${}^Gg$  is infinite. It follows that  $G$  is icc.

Conditions (i) and (ii)  $\implies G$  is icc.

Suppose that condition (i) and (ii) are both satisfied. Obviously each element  $(\phi, 1)$  with  $\phi \neq \varepsilon$  has an infinite conjugacy class. Let  $g = (\phi, q)$  with  $q \neq 1$ . If  $q \notin FC(Q)$  then  $Qg$  and  ${}^Gg$  are infinite. If  $q \in FC(Q)$ , condition (i) implies that  $q$  does not fix some element

$y \in \Omega$ . Consider for any  $d \in D$  the conjugate  $g(d) = (\eta_d^y, 1)^{-1}(\phi, q)(\eta_d^y, 1)$  of  $g$ . If  $y$  does not lie in the support  $Supp(\phi)$  of  $\phi$ , then  $g(d) = (\phi\eta_{d-1}^y\eta_d^{qy}, q)$ . If  $y \in Supp(\phi)$ , say  $\phi = \phi_0\eta_c^y$  and  $y \notin Supp(\phi_0)$ , then  $g(d) = (\phi_0\eta_{d-1}^y\eta_d^{qy}, q)$ . In any case, since  $qy \neq y$  all  $g(d)$  for  $d \in D$  are distinct. Since  $D$  is icc  $D$  is infinite, and so  $g$  has an infinite conjugacy class. Hence  $G$  is icc.  $\square$

From theorem 2.1 one immediately deduces the stability of icc property by wreath product.

**Corollary 2.2.** *The icc property is stable by wreath products: any complete (respectively restricted) wreath product of icc groups is icc.*

### 3. SEMI-DIRECT PRODUCT

Throughout this section the group  $G$  stands for the *semi-direct product* (or *split extension*)  $G = K \rtimes_{\theta} Q$  with normal factor  $K$ , retract factor  $Q$  and associated homomorphism  $\theta : Q \rightarrow Aut(K)$ ;  $K$  and  $Q$  will be seen as subgroups of  $G$ . With this notation, for any  $k \in K$ ,  $q \in Q$ ,  $qkq^{-1} = \theta(q)(k)$  in  $G$ . Let  $\pi : G \rightarrow Aut(K)$  be the homomorphism defined by  $\forall g \in G, k \in K$ ,  $\pi(g)(k) = gkg^{-1}$ ; it extends on  $G$  both  $\theta$  and the natural homomorphism  $\pi_K : K \rightarrow Inn(K)$ , that is the diagram below commutes (the two horizontal sequences being exact).

$$\begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q \longrightarrow 1 \\ & & \downarrow \pi_K & & \downarrow \pi & & \downarrow \theta \\ 1 & \longrightarrow & Inn(K) & \longrightarrow & \pi(G) & \longrightarrow & \theta(Q) \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & Aut(K) \end{array}$$

We shall write in the following  $\theta_q$  and  $\pi_g$  instead of  $\theta(q)$  and  $\pi(g)$ . We will denote by  $\Phi : FC(Q) \rightarrow Out(K)$  the homomorphism induced by  $\theta : Q \rightarrow Aut(Q)$ , *i.e.* which makes the following diagram commute:

$$\begin{array}{ccc} Q & \xrightarrow{\theta} & Aut(K) \\ \uparrow & & \downarrow \\ FC(Q) & \xrightarrow{\Phi} & Out(K) \end{array}$$

**3.1. Statement of the results.** We give four equivalent characterizations of semi-direct products with infinite conjugacy classes:

**Theorem 3.1** (semi-direct product icc; version I). *Let  $G = K \rtimes_{\theta} Q \neq 1$  be a split extension. Then  $G$  is not icc if and only if one of the following conditions is satisfied:*

- (i)  *$FC(K)$  contains a subgroup  $N \neq \{1\}$  normal in  $K$  preserved under the action of  $\theta(Q)$  and such that either  $N$  is finite, or  $N \approx \mathbb{Z}^n$  and the induced homomorphic image of  $\theta(Q)$  in  $GL(n, \mathbb{Z})$  is finite.*
- (ii)  *$\Phi : FC(Q) \rightarrow Out(K)$  is non injective, and  $\ker \Phi$  contains  $q \neq 1$  such that  $\theta_q(x) = kxk^{-1}$ , for some  $k \in K$  with finite  $\theta(Q)$ -orbit.*

The result becomes by mean of the finite  $\theta(Q)$ -orbits:

**Theorem 3.2** (semi-direct product icc; version II). *Let  $G = K \rtimes_{\theta} Q \neq 1$  be a split extension. Then  $G$  is not icc if and only if  $K$  or  $Q$  is not icc and one of the following conditions is satisfied:*

- (i)  $\theta(Q)$  has a finite orbit in  $FC(K) \setminus \{1\}$ ,
- (ii.a)  $\theta(Q)$  has a finite orbit in  $\pi_K^{-1}(\theta(FC(Q))) \setminus \{1\}$ ,
- (ii.b) the restricted homomorphism  $\theta : FC(Q) \longrightarrow Aut(K)$  is non injective.

*Remarks.*

- In Theorem 3.2 one can change in any of the conditions (i) and (ii.a) the word "in" by "that intersects". It follows from the facts that  $FC(K), FC(Q)$  are characteristic in  $K, Q$  and  $Inn(K)$  is normal in  $Aut(K)$ .
- In particular when  $K \setminus \{1\}$  contains only infinite  $\theta(Q)$ -orbits then  $G$  is icc if and only if  $\theta : FC(Q) \longrightarrow Aut(K)$  is injective.

The following theorems differ from Theorem 3.1 by reformulating condition (ii). Define the homomorphism  $\Pi : G \longrightarrow Aut(G)$  by,  $\forall q \in Q, x \in G, \Pi(q)(x) = {}^qx$ ; it makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\Pi} & Aut(G) \\ & \searrow \pi & \downarrow \\ & & Aut(K) \end{array}$$

**Theorem 3.3** (semi-direct product icc; version III). *Let  $G = K \rtimes_{\theta} Q \neq 1$  be a split extension. Then  $G$  is not icc if and only if one of the following conditions is satisfied:*

- (i)  $FC(K)$  contains a subgroup  $N \neq \{1\}$  normal in  $K$  preserved under the action of  $\theta(Q)$  and such that either  $N$  is finite, or  $N \approx \mathbb{Z}^n$  and the induced homomorphic image of  $\theta(Q)$  in  $GL(n, \mathbb{Z})$  is finite.
- (ii)  $FC(Q)$  contains a non-trivial subgroup  $C$  normal in  $Q$ , either finite or  $C \approx \mathbb{Z}^n$ , and there exists an homomorphic cross-section  $s : C \longrightarrow G$  such that  $K \rtimes_{\theta} C = K \times s(C)$  and  $s(C)$  is preserved with only finite  $\Pi(Q)$ -orbits.

The next formulation involves cohomology of groups. The key point is that in Theorem 3.1.(ii) the condition " $\theta_q(x) = {}^kx$  for some  $k \in K$  with finite  $\theta(Q)$ -orbit" does depend on the choice of  $k \in K$  such that  $\theta(q)(x) = {}^kx$ , i.e. on the choice of an element in  $k_0Z(K)$  for some  $k_0 \in K$  such that  $\forall x \in K, \theta(q)(x) = {}^{k_0}x$ . The formulation of Theorem 3.4.(ii) prevents such a choice, by mean of a cohomological condition. Furthermore among those last four results, Theorem 3.4 is the one which admits a generalization for general extension.

We need to introduce the *collective character* associated to the extension,  $\Theta : Q \longrightarrow$

$Out(K)$  which makes the following diagram commute:

$$\begin{array}{ccc} Q & \xrightarrow{\theta} & Aut(K) \\ & \searrow \Theta & \downarrow \\ & & Out(K) \end{array}$$

The following notion will be detailed in section 3.2.4. It will be generalized to any extension of group in section 4, so that the reader may omit the details in §3.2.4 and skip to the most general and technical §4.1.

**Proposition-Definition 3.1.** *Any element  $q \in \ker \Theta$  defines an element  $[q]$  in the first cohomology group  $H^1(Z_Q(q), Z(K))$ . If  $\theta_q(x) = k x k^{-1}$  for some  $k \in K$ , then  $[q]$  is represented by the 1-cocycle:*

$$\begin{array}{ccc} Z_Q(q) & \longrightarrow & Z(K) \\ u & \longrightarrow & [k^{-1}, u] \end{array}$$

Indeed  $[q]$  does not depend on the choice of  $k \in K$  such that  $\theta_q(x) = {}^k x$ .

We'll see further that for any  $q \in FC(Q) \setminus \{1\}$  the cohomological condition  $[q] = 0$  implies the existence of some  $k \in K$  with finite  $\theta(Q)$ -orbit such that  $\theta_q(x) = {}^k x$ , that is condition (ii) of Theorem 3.1. Reciprocally condition (ii) of Theorem 3.1 implies that either  $\exists q \in FC(Q) \setminus \{1\}$  such that  $[q] = 0$  or condition (i) of Theorem 3.1 is satisfied.

**Theorem 3.4** (semi-direct product icc; version IV). *Let  $G = K \rtimes_{\theta} Q \neq 1$  be a split extension. Then  $G$  is not icc if and only if one of the following conditions is satisfied :*

- (i)  *$FC(K)$  contains a subgroup  $N \neq \{1\}$  normal in  $K$  preserved under the action of  $\theta(Q)$  and such that either  $N$  is finite, or  $N \approx \mathbb{Z}^n$  and the induced homomorphic image of  $\theta(Q)$  in  $GL(n, \mathbb{Z})$  is finite.*
- (ii)  *$\Phi : FC(Q) \longrightarrow Out(K)$  is non injective, and  $\ker \Phi$  contains  $q \neq 1$  which defines an element  $[q] = 0$  in  $H^1(Z_Q(q), Z(K))$ .*

See §3.4.2 (or skip to §5.9) for more on this condition (ii) and for a noteworthy particular case.

**Corollary 3.1.** *The icc property is stable by semi-direct product: whenever  $K, Q$  are icc,  $K \rtimes Q$  is icc.*

**Corollary 3.2.** *If  $K$  and  $\theta(Q)$  are icc then  $K \rtimes_{\theta} Q$  is icc.*

*Example.* Suppose  $K$  contains a characteristic subgroup with finite automorphism group, then  $G$  is not icc: e.g.  $K \neq \{1\}$  is finite,  $K$  is virtually  $\mathbb{Z}$ , or  $Tor(K)$  (respectively  $Tor(Z(K))$  or  $Tor(FC(K))$ ) is a non trivial finite group (in each case condition (i) of Theorems 3.1-3.4 is satisfied.)

**3.2. Proofs of the results.** This section is devoted to the proofs of the theorems enonced in section 3.1.

**3.2.1. Proof of Theorem 3.1.** We first prove the sufficient part of the assumption, that is, if either condition (i) or condition (ii) is satisfied, then  $G$  is not icc.

**Fact 1.** *Condition (i) implies that  $G$  is not icc.*

*Proof of the fact 1.* Suppose the condition (i) is satisfied. By hypothesis  $N$  is preserved under the action of  $\pi_K(K)$  (respectively  $\theta(Q)$ ) with only finite orbits. Since the conjugacy class in  $K$  (respectively in  $G$ ) of an element of  $K$  is its orbit under the action of  $\pi_K(K)$  (respectively of  $\pi(G)$ ), and since  $\pi(G)$  equals the product  $\theta(Q)\pi_K(K)$ , each element of  $N$  has a finite conjugacy class in  $G$ . As  $N \neq \{1\}$ ,  $G$  is not icc.

**Fact 2.** *Condition (ii) implies that  $G$  is not icc.*

*Proof of the fact 2.* Suppose the condition (ii) is satisfied; let  $\omega = k^{-1}q \neq 1$ , so that  $Z_G(\omega) \supset K$ . Let  $\text{Stab}_{\theta(Q)}(k)$  denotes the stabilizer of  $k$  in  $\theta(Q)$ ; since it has a finite index in  $\theta(Q)$ , then  $Q_0 = \theta^{-1}(\text{Stab}_{\theta(Q)}(k))$  has a finite index in  $Q$ . Hence  $Q_1 = Z_Q(q) \cap Q_0$  also has a finite index in  $Q$ . Then for any  $u \in Q_1$ ,  ${}^u\omega = u k^{-1}q u^{-1} = \theta_u(k^{-1}) {}^uq = k^{-1}q = \omega$ ; hence  $Z_G(\omega) \supset Q_1$ . It follows that  $Z_G(\omega)$  contains  $K \rtimes Q_1$  and hence has a finite index in  $G$ , so that  $G$  is not icc.

We now prove the necessary part of the assumption, that is, if  $G$  is not icc then either condition (i) or condition (ii) is satisfied. Let  $G$  be not icc : since  $G \neq \{1\}$ , there exists  $u \neq 1$  in  $G$  such that  ${}^G u$  is finite.

**Fact 3.** *If  $K$  contains  $u \neq 1$  with  ${}^G u$  finite, then condition (i) is satisfied.*

*Proof of the fact 3.* Let  $N'$  be the subgroup of  $K$  finitely generated by the set  ${}^G u$ . Then  $N'$  is preserved under the action of  $\pi(G)$ , and in particular is normal in  $K$ . Since any element of  ${}^G u$  has a finite orbit under  $\pi(G)$ ,  $N'$  contains only finite  $\pi(G)$ -orbits. In particular  $N'$  is a finitely generated  $FC$ -group. It follows that  $\text{Tor}(N')$  is a finite characteristic subgroup of  $N'$  and  $N'/\text{Tor}(N')$  is free abelian with finite rank (cf. [Ne]). Then one obtains a normal subgroup  $N$  of  $K$  in  $FC(K)$  with only finite  $\theta(Q)$ -orbits by : if  $\text{Tor}(N') \neq \{1\}$  then  $N = \text{Tor}(N')$  and otherwise  $N = N' = \mathbb{Z}^n$ . In the latter case it naturally gives rise to an homomorphism from  $\theta(Q)$  to  $GL(n, \mathbb{Z})$  whose image has only finite orbits in  $\mathbb{Z}^n$ . It's an easy exercise that infinite subgroups of  $GL(n, \mathbb{Z})$  have infinite orbits on  $\mathbb{Z}^n$  so that this image is a finite subgroup of  $GL(n, \mathbb{Z})$ . Hence condition (i) is satisfied.

**Fact 4.** *If  $G \setminus K$  contains  ${}^G u$  finite, then either condition (i) or (ii) is satisfied.*

*Proof of the fact 4.* Let  $u = k^{-1}q$  for some  $k \in K$  and  $q \neq 1$  lying in  $Q$ , such that  $Z_G(u)$  has a finite index in  $G$ . Necessarily  $q$  lies in  $FC(Q)$ , for  ${}^Q q$  is the image of  ${}^G u$  under the projection of  $G$  onto  $Q$ .

Let  $h \in K$  and  $\omega = [u, h] \in K$ ; both  $Z_G(u)$  and  $Z_G(hu^{-1}h^{-1})$  have a finite index in  $G$  and their intersection lies in  $Z_G(\omega)$ , so that  $\omega$  is an element of  $K$  having a finite conjugacy class in  $G$ . If  $\omega \neq 1$ , it follows from the fact 3 that condition (i) is satisfied. So we suppose in the following that for any  $h \in K$ ,  $[u, h] = 1$ , so that  $\pi_u$  is the identity on  $K$ . Hence  $\theta_q$  is inner, for any  $x \in K$ ,  $\theta_q(x) = {}^k x$ .

Now let  $Q_0 = Z_G(u) \cap Z_Q(q)$ ,  $Q_0$  is obviously contained in  $Z_Q(k)$ , so that  $\theta(Q_0)$  is contained in  $\text{Stab}_{\theta(Q)}(k)$ . Since  $Q_0$  has a finite index in  $Z_Q(q)$ , it also has a finite index in  $Q$ , and then  $\text{Stab}_{\theta(Q)}(k)$  has a finite index in  $\theta(Q)$ , so that  $k$  has a finite  $\theta(Q)$ -orbit. Hence condition (ii) is satisfied.  $\square$

**3.2.2. Proof of Theorem 3.2.** We prove the equivalence with Theorem 3.1. Condition (i) of Theorem 3.1 obviously implies that  $FC(K) \setminus \{1\}$  contains a finite  $\theta(Q)$ -orbit. The



converse is also true. For, if  $FC(K)$  contains a non-trivial finite  $\theta(Q)$ -orbit  $O \neq \{1\}$ , the union of conjugates of  $O$  in  $K$  is finite and preserved under  $\pi(G)$ , so that for  $k_0 \in O$ ,  ${}^G k_0$  is finite,  $k_0 \neq 1$ , and condition (i) of theorem 3.1 follows from the fact 3 in the proof of theorem 3.1.

Disjunction of (ii.a) and (ii.b) is just a rephrasing of condition (ii) of theorem 3.1 where in (ii.a) the finite  $\theta(Q)$ -orbit is the one of  $k \neq 1$  and (ii.b) occurs when  $k = 1$ .  $\square$

**3.2.3. Proof of Theorem 3.3.** In order to prove the theorem, we first describe a phenomenon which appears when condition (ii) of theorem 3.1 is satisfied while condition (i) is not and  $\theta : FC(Q) \rightarrow \text{Aut}(K)$  is injective. The lemma 3.1 below provides a fifth characterization of icc property that we have nevertheless chosen not to emphasize.

**Definition.** Let  $G = K \rtimes_{\theta} Q$ ; We say that  $C' \rtimes C$  is a *twin FC-subfactor* of  $G$  when :

- $C'$  is a subgroup of  $K$ ,  $C' \cap FC(K) = \{1\}$ ,
- $C'$  is  $\theta(Q)$ -stable with only finite  $\theta(Q)$ -orbits,
- $C$  is a normal subgroup of  $Q$ ,  $C \subset FC(Q)$ ,
- $\pi$  and  $\theta$  are injective respectively on  $C'$  on  $C$  and  $\pi(C') = \theta(C)$ ,  
(so that  $\pi|_{C'}^{-1} \circ \theta|_C : C \rightarrow C'$  is an isomorphism),
- $C \neq \{1\}$  (and so  $C'$ ) is either finite or  $\mathbb{Z}^n$ .

A twin FC subfactor is either  $\mathbb{Z}^n \times \mathbb{Z}^n$  or  $C \rtimes_{\text{Inn}(C)} C$ , for some finite group  $C$ . It is  $\Pi(Q)$ -stable with only finite orbits, and  $\theta|_C^{-1} \circ \pi|_{C'}$  sends isomorphically the normal factor on the retract one.

**Lemma 3.1.** *Let  $G = K \rtimes_{\theta} Q \neq 1$ ; Then  $G$  is not icc if and only if either condition (i) or at least one of the following conditions is satisfied :*

- (ii.a) *the restricted homomorphism  $\theta : FC(Q) \rightarrow \text{Aut}(K)$  is non injective,*
- (ii.b)  *$G$  contains a twin FC-subfactor.*

*Proof.* We first consider the sufficient part of the assumption. That condition (i) implies that  $G$  is not icc follows from Theorem 3.1; obviously condition (ii.a) also implies that  $G$  is not icc. If  $G$  contains a twin FC-subfactor  $C' \rtimes_{\text{Inn}(C')} C$ , then condition (ii) of Theorem 3.1 is satisfied with  $q$  being any non-trivial element of  $C$  and  $k = \pi|_{C'}^{-1} \circ \theta|_C(q)$ , so that  $G$  is not icc.

We now prove the necessary part of the assumption. We suppose in the following that  $G$  is not icc while it satisfies neither condition (i) nor condition (ii.a) and prove that condition (ii.b) must be satisfied.

With Theorem 3.1, there exists  $q \neq 1$  in  $FC(Q)$  and  $k \neq 1$  in  $K$ , such that  $\theta_q(x) = {}^k x$  and  $\text{Stab}_{\theta(Q)}(k)$  has a finite index in  $\theta(Q)$ . Let  $C_Q$  be the subgroup of  $FC(Q)$  finitely generated by  ${}^Q q$ ;  $C_Q$  is a non-trivial FC-group normal in  $Q$ . Let  $Q_1 = \theta^{-1}(\text{Stab}_{\theta(Q)}(k))$ , it has a finite index in  $Q$ . Note that  $\text{Stab}_{\theta(Q)}(k)$  is included in  $Z_{\theta(Q)}(\theta_q)$ , for let  $\varphi \in \text{Stab}_{\theta(Q)}(k)$  and  $x \in K$ ,  $\varphi \circ \theta_q \circ \varphi^{-1}(x) = \varphi(k \varphi^{-1}(x) k^{-1}) = k x k^{-1} = \theta_q(x)$ . Hence if one would have  $Q_1 \not\subset Z_Q(q)$ , there would exist  $p \in Q$  such that  $[p, q] \neq 1$  and  $\theta([p, q]) = 1$ , which would imply condition (ii.a), which has been excluded. Hence,  $Q_1 \subset Z_Q(q)$ , so that for  $q_0 = 1, q_1, \dots, q_p$  a set of representatives of  $Q/Q_1$ ,  $C_Q$  is generated by the finite family  $q, {}^{q_1}q, \dots, {}^{q_p}q$ .

Let  $k_i = \theta_{q_i}(k)$  for  $i = 1, \dots, p$ , then  $\{k, k_1, \dots, k_p\}$  is the  $\theta(Q)$ -orbit of  $k$  and moreover  $\theta_{q_i} \circ \theta_q \circ \theta_{q_i}^{-1}(x) = {}^{k_i}x$ . Let  $C_K$  be the subgroup of  $K$  generated by  $k, k_1, \dots, k_p$ ;  $C_K$  is preserved under  $\theta(Q)$  and contains only finite  $\theta(Q)$ -orbits. An element in  $C_K \cap FC(K)$  would have a finite conjugacy class in  $G$ , so that  $C_K \cap FC(K) = \{1\}$  because otherwise as in fact 3 in the proof of Theorem 3.1 condition (i) would follow.

By construction,  $\pi(C_K) = \theta(C_Q)$ . Each element of  $\text{Ker } \pi|_{C_K}$  has a finite conjugacy class in  $G$  so that  $\pi$  must be injective on  $C_K$  because otherwise, as in fact 3 in the proof of Theorem 3.1, condition (i) would follow. Moreover  $\theta$  is injective on  $C_Q$  because otherwise condition (ii.a) would be satisfied. Hence  $\theta^{-1} \circ \pi|_{C_K} : C_K \rightarrow C_Q$  is an isomorphism. Now  $C_Q \neq \{1\}$  is a finitely generated  $FC$ -group and hence with [Ne] either  $\text{Tor}(C_Q) \neq \{1\}$  is a finite normal subgroup in  $Q$ , in which case let  $C = \text{Tor}(C_Q)$ , or  $C_Q \approx \mathbb{Z}^n$ , in which case let  $C = C_Q$ . Let  $C'$  denotes  $\pi|_{C_K}^{-1} \circ \theta|_{C_Q}(C)$ , then  $C' \rtimes_\theta C$  is a twin  $FC$ -subfactor in  $G$ .  $\square$

We now turn to the proof of Theorem 3.3.

*Proof of Theorem 3.3.* We proceed in several steps.

*Step 1. Condition (i)  $\implies G$  not icc.* Simply follows from Theorem 3.1.(i).

*Step 2. Condition (ii)  $\implies G$  not icc.* Consider a non-trivial element  $\omega$  lying in  $s(C)$ . Then  $Z_G(\omega) \supset K$  so that  ${}^G\omega$  is the orbit of  $\omega$  under  $\Pi(Q)$ , and hence is finite.

*Step 3.  $\theta : FC(Q) \rightarrow \text{Aut}(K)$  non injective  $\implies$  condition (ii).* Let  $q \in FC(Q) \cap \ker \theta$ ;  ${}^Qq$  generates a finitely generated  $FC$ -group  $D \neq \{1\}$ , normal in  $Q$ , lying in  $FC(Q) \cap \ker \theta$ . Applying [Ne],  $D$  contains a normal subgroup  $C \neq \{1\}$  either finite or  $\mathbb{Z}^n$ , and condition (ii) is satisfied with  $s : C \hookrightarrow G$  the identity.

*Step 4.  $G$  not icc  $\implies$  condition (i) or (ii).* We suppose that  $G$  is not icc, that  $\theta : FC(Q) \rightarrow \text{Aut}(K)$  is injective and that condition (i) is not satisfied and prove that condition (ii) is satisfied. According to the lemma 3.1,  $G$  contains an  $FC$ -subfactor  $C' \rtimes_{\text{Inn}(C')} C$ . Denote by  $\varphi = \pi|_{C'}^{-1} \circ \theta|_C$ ; then  $\varphi : C \rightarrow C'$  is an isomorphism. Consider  $s : C \rightarrow G$  defined by  $\forall \gamma \in C$ ,  $s(\gamma) = \varphi(\gamma)^{-1} \gamma$ . Then  $s$  is an homomorphism since:

$$\begin{aligned} s(\gamma_1) s(\gamma_2) &= \varphi(\gamma_1)^{-1} \gamma_1 \varphi(\gamma_2)^{-1} \gamma_2 \\ &= \varphi(\gamma_1)^{-1} \gamma_1 \varphi(\gamma_2)^{-1} \gamma_1 \gamma_2 \\ &= \varphi(\gamma_1)^{-1} \varphi(\gamma_1) \varphi(\gamma_2)^{-1} \gamma_1 \gamma_2 \\ &= \varphi(\gamma_1 \gamma_2)^{-1} \gamma_1 \gamma_2 \\ &= s(\gamma_1 \gamma_2) \end{aligned}$$

which is injective since  $K \cap Q = \{1\}$ , and obviously a cross section of  $K \rtimes_\theta C$ ; moreover  $K \rtimes_\theta C = K \times s(C)$  since  $\forall \gamma \in C$ ,  $\pi(s(\gamma)) = \pi_K(\varphi(\gamma)) \circ \theta(\gamma) = \text{Id}_K$ . Since  $\Pi(Q)$  preserves with only finite orbits both  $C$  and  $C'$ ,  $s(C)$  is also preserved with only finite  $\Pi(Q)$ -orbits. Hence condition (ii) is satisfied.  $\square$

**3.2.4. Proof of Theorem 3.4.** Throughout the section, we fix the semi-direct product  $G = K \rtimes_\theta Q$  and an element  $q \in Q$  such that  $\theta_q \in \text{Inn}(K)$ ,

$$\exists k \in K, \forall x \in K, \quad \theta_q(x) = k x k^{-1}.$$

**Proposition 3.1.** *If  $p \in Q$  is such that  $[p, q] = 1$  then  $k^{-1} \theta_p(k) \in Z(K)$ .*

*Proof.* Since  $[p, q] = 1$  one has  $\theta_q = \theta_p^{-1} \circ \theta_q \circ \theta_p$ . Hence  $\forall x \in K$ ,  $k x k^{-1} = \theta_p^{-1}(k \theta_p(x) k^{-1}) = \theta_p^{-1}(k) x \theta_p^{-1}(k^{-1})$ , so that  $\forall x \in K$ ,  $k^{-1} \theta_p(k)$  commute with  $x$ , that is  $k^{-1} \theta_p(k) \in Z(K)$ .  $\square$

Now define the map:

$$\begin{aligned} d_q : \quad Z_Q(q) &\longrightarrow Z(K) \\ u &\longrightarrow d_q(u) = [k^{-1}, u] \end{aligned}$$

**Proposition 3.2.** *The map  $d_q$  is a 1-cocycle (or crossed homomorphism).*

*Proof.* We only need to verify that  $\forall u, v \in Z_Q(q)$ ,  $d_q(uv) = d_q(u)^u d_q(v)$ .

$$d_q(uv) = [k^{-1}, uv] = k^{-1}uvkv^{-1}u^{-1} = k^{-1}ukd_q(v)u^{-1} = k^{-1}uku^{-1}^u d_q(v) = d_q(u)^u d_q(v) .$$

□

The cocycle  $d_q$  depends on the choice of  $k$  such that  $\forall x \in K$ ,  $\theta_q(x) = {}^k x$ , while its image in  $H^1(Z_Q(q), Z(K))$  does not.

**Proposition 3.3.** *The 1-cocycle  $d_q$  defines an element  $[q]$  of  $H^1(Z_Q(q), Z(K))$  which only depends on  $q$ .*

*Proof.* Let  $k' \in kZ(K)$  and consider the 1-cocycle  $d'_q$  defined by  $d'_q(x) = [k'^{-1}, x]$ . Let  $[q]$ ,  $[q']$  be the class respectively of  $d_q$  and  $d'_q$  in  $H^1(Z_Q(q), Z(K))$ . Since  $k' = kz$  for some  $z \in Z(K)$ , one has:

$$d'_q(x) = z^{-1}k^{-1}xkzx^{-1} = z^{-1}xz k^{-1}xkx^{-1} = z^{-1}xz d_q(x) .$$

Now  $x \longrightarrow z^{-1}xz$  is a 1-coboundary, so that  $[q] = [q']$ . □

The above proves the proposition-definition 3.1. We now turn to the proof of theorem 3.4.

*Proof of Theorem 3.4.* We prove the equivalence between condition (ii) and condition (ii) of Theorem 3.1. We proceed in two steps.

*Step 1. Condition (ii) of Theorem 3.1  $\implies$  condition (i) or (ii).*

Let  $q \in FC(Q)$  such that  $\theta_q(x) = kxk^{-1}$  for some  $k$  with finite  $\theta_q$ -orbit. Then  $Q_0 = \theta^{-1}(Stab_{\theta(Q)}(k))$  has a finite index in  $Q$ , and let  $Z_0 = Q_0 \cap Z_Q(q)$ ;  $Z_0$  has a finite index in  $Q$ .

Suppose  $Z_Q(q) \setminus Z_0 \neq \emptyset$  and let  $u \in Z_Q(q) \setminus Z_0$ ; then there exists  $z \neq 1$  in  $Z(K)$  such that  $uku^{-1} = kz$  (proposition 3.1). Then  $Z_G(z) \supset K$ , and since  $z = k^{-1}uku^{-1}$ ,  $Z_G(z) \supset Q_0 \cap uQ_0u^{-1}$ . Hence  $Z_G(z)$  has a finite index in  $G$ ; following the fact 3 in the proof of Theorem 3.1 condition (i) is satisfied.

Suppose now on the contrary that  $Z_0 = Z_Q(q)$ . For any  $u \in Z_Q(q)$ ,  $\theta(u) \in Stab_{\theta(Q)}(k)$  so that the  $d_q(u) = [k^{-1}, u] = 1$  and  $[q] = 0$  in  $H^1(Z_Q(q), Z(K))$ ; condition (ii) is satisfied.

*Step 2. Condition (ii)  $\implies$  condition (ii) of Theorem 3.1.*

Let  $q \in FC(Q)$  and  $k \in K$  such that  $\forall x \in K$ ,  $\theta_q(x) = kxk^{-1}$ , and suppose that  $[q] = 0$  in  $H^1(Z_Q(q), Z(K))$ . There exists  $z \in Z(K)$  such that  $\forall u \in Z_Q(q)$ ,  $k^{-1}uku^{-1} = z^u z^{-1}$  and it follows that  $[u, kz] = 1$ :

$$\begin{aligned} k^{-1}uk u^{-1} &= z^u z^{-1} \\ \iff z^{-1}k^{-1}uk u^{-1}u z &= 1 \\ \iff (kz)^{-1}uk z u^{-1} &= 1 \end{aligned}$$

Let  $k' = kz$ , then  $\theta_q(x) = k'xk'^{-1}$  and since  $Z_Q(q)$  has a finite index in  $Q$ ,  $k'$  has a finite  $\theta(Q)$ -orbit. Condition (ii) of Theorem 3.1 is satisfied. □

**3.3. Examples.** We now consider two examples. The first example shows that condition (ii) of theorem 3.1 cannot in general be weakened into the condition that  $\Phi : FC(Q) \longrightarrow Out(K)$  is non injective.

3.3.1. *Example 1.* Consider:

$$K = \langle a_1, a_2, k_1, k_2 \mid [a_1, a_2], [a_i, k_j], i, j = 1, 2 \rangle \approx (\mathbb{Z} \oplus \mathbb{Z}) \times F_2$$

$$A = \langle a_1, a_2 \rangle_K \approx \mathbb{Z} \oplus \mathbb{Z} \subset K$$

$$Q = \langle q_1, q_2 \mid [q_1, q_2] \rangle \approx \mathbb{Z} \oplus \mathbb{Z}$$

Let  $\theta_1 \in \text{Inn}(K)$ , s.t.  $\forall x \in K, \theta_1(x) = k_1 x$ ;  $\theta_1$  fixes  $A$  pointwise. Let  $\theta_2 \in \text{Aut}(K)$ , s.t.  $\theta_2$  is anosov on  $A$ , and  $\theta_2(k_2) = k_2$ ,  $\theta_2(k_1) = k_1 \alpha$  for some  $\alpha$  lying in  $A$ . So defined,  $\theta_1$  and  $\theta_2$  commute, so that the map sending  $q_1$  to  $\theta_1$  and  $q_2$  to  $\theta_2$  extends to an homomorphism  $\theta : Q \longrightarrow \text{Aut}(K)$ ; moreover  $\theta$  is injective.

In this example we suppose that  $\alpha \neq 1$ . Consider  $G = K \rtimes_{\theta} Q$ ; we show that  $G$  is icc despite that  $\Phi : FC(Q) \longrightarrow \text{Out}(K)$  is non injective. For any non-trivial  $x \in K$ ,  ${}^G x$  is infinite, so that in particular condition (i) of theorem 3.1 is not satisfied. If condition (ii) would be satisfied, it would follow that for some  $n \geq 1$ ,  $k_1^n$  would have a  $\theta(Q)$ -finite orbit. We show that this cannot arise.

Consider  $\theta_2 \in \theta(Q)$ ,  $\theta_2(k_1) = k_1 \alpha$ ,  $\alpha \neq 1 \in A$ , so that for any  $p \geq 1$ ,

$$\theta_2^p(k_1^n) = k_1^n \alpha^n \theta_2(\alpha^n) \theta_2^2(\alpha^n) \cdots \theta_2^{p-1}(\alpha^n)$$

Let  $\phi_p : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$  be the map defined by  $\phi_p(x) = x \theta_2(x) \theta_2^2(x) \cdots \theta_2^{p-1}(x)$ ;  $\phi_p$  turns out to be an homomorphism. Let  $M_{\theta} \in SL(2, \mathbb{Z})$  be the matrix associated with  $\theta_2$ ; it has two distinct irrational eigen values  $\lambda_1, \lambda_2$ . Let  $M_p$  be the matrix associated with  $\phi_p$ . Then  $M_p = Id + M_{\theta} + M_{\theta}^2 + \cdots + M_{\theta}^{p-1}$ .  $M_p$  has two eigen values :  $l_i = 1 + \lambda_i + \lambda_i^2 + \cdots + \lambda_i^{p-1}$ ,  $i = 1, 2$ . They must be both non null because otherwise  $\lambda_i^p = 1$  which contradicts that  $M_{\theta}$  is anosov. Hence, for any  $p \geq 1$ ,  $\phi_p$  is injective. Since for any  $n \geq 1$ ,  $\theta_2^p(k_1^n) = k_1^n \phi_p(\alpha^n)$ , with  $\alpha^n \neq 1 \in A$ , the  $\theta(Q)$ -orbit of  $k_1^n$  is infinite, so that condition (ii) is not satisfied. With theorem 3.1,  $G$  is icc, despite that the homomorphism  $\Phi : FC(Q) \longrightarrow \text{Out}(K)$  is non injective.

3.3.2. *Example 2.* We consider the same example as before despite we suppose here that  $\alpha = 1$ . Condition (i) of Theorem(s) 3.1 (to 3.4) is not satisfied since  $FC(K) = A$  contains only infinite  $\theta(Q)$ -orbits. Nevertheless  $G$  is not icc; let's verify that conditions (ii) of Theorems 3.1 to 3.4 are all satisfied:

- $\theta_{q_1}(x) = k_1 x k_1^{-1}$  and  $\theta(Q)$  fixes  $k_1$ : Theorem 3.1.(ii) is satisfied;
- $\{k_1\}$  is a finite  $\theta(Q)$ -orbit lying in  $\pi_K^{-1}(\theta(FC(Q)))$ : Theorem 3.2.(ii) is satisfied;
- let  $C' \subset K$ ,  $C \subset Q$  be generated respectively by  $k_1$  and  $q_1$ , then  $C' \rtimes_{\theta} C = \mathbb{Z} \times \mathbb{Z}$  is a twin  $FC$ -subfactor in  $G$ : Lemma 3.1.(ii.b) is satisfied;
- let  $s : C \longrightarrow G$  s.t.  $s(q_1) = k_1^{-1} q_1$ , then  $s(C) = \mathbb{Z}$  contains only finite  $\Pi(Q)$ -orbits and  $K \rtimes C = K \times s(C)$ : Theorem 3.3.(ii) is satisfied.
- $Z_Q(q_1) = Q \approx \mathbb{Z} \oplus \mathbb{Z}$ ,  $Z(K) = A \approx \mathbb{Z} \oplus \mathbb{Z}$  and  $[q_1] = 0$  in  $H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$  for it is represented by  $d_{q_1}$  which is a crossed homomorphism identically null: Theorem 3.4.(ii) is satisfied.

3.4. **Particular cases.** We now consider some particular cases, where we suppose some additional hypothesis on the groups involved and for which the results take a more concise form. Numerous other particular cases will be proved more generally in §5.

3.4.1. In case  $Q = \text{Aut}(K)$ .

In this part we consider the semi-direct product  $G = K \rtimes Q$  where  $Q = \text{Aut}(K)$  and  $\theta : Q \rightarrow \text{Aut}(K)$  is the identity.

**Proposition 3.4** (in case  $Q = \text{Aut}(K)$ ). *Let  $G = K \rtimes \text{Aut}(K)$ , then  $G$  is icc if and only if  $\text{Aut}(K)$  has no finite orbit in  $K \setminus \{1\}$ .*

*Proof.* According to theorem 3.2, if  $\text{Aut}(K)$  has no finite orbit in  $K \setminus \{1\}$  then  $G$  is icc. Reciprocally, let  $\{k_1, \dots, k_n\}$  be a finite orbit of  $\text{Aut}(K)$  in  $K$ . Denote by  $\theta_1, \dots, \theta_n \in \text{Inn}(K)$  such that for  $i = 1, \dots, n$ , and  $x \in K$ ,  $\theta_i(x) = k_i x k_i^{-1}$ . Given  $\theta \in \text{Aut}(K)$ ,  ${}^\theta\theta_i$  is the element of  $\text{Inn}(K)$  defined  $\forall x \in K$ ,  ${}^\theta\theta_i(x) = \theta(k_i) x \theta(k_i)^{-1}$  so that  $\theta_1, \dots, \theta_n$  is a finite conjugacy class in  $\text{Aut}(K)$  and condition (ii.a) of theorem 3.2 is satisfied; hence  $G$  is not icc.  $\square$

*Example.* Let  $G = A \rtimes_\theta \text{Aut}(A)$  where  $A$  is a finitely generated abelian group. Then  $G$  is icc if and only if  $A$  is torsion-free and  $\text{rank}(A) > 1$ .

The same argument shows that:

**Proposition 3.5.** *If  $\text{Aut}(K)$  is icc then  $K \rtimes \text{Aut}(K)$  is also icc.*

We can also emphasize the following:

**Proposition 3.6.** *In the assumptions of theorems 3.1 to 3.4, if moreover the  $\theta(Q)$ -extension  $\pi(G)$  of  $\text{Inn}(K)$  splits, i.e.  $\pi(G) = \text{Inn}(K) \rtimes \theta(Q)$ , then condition (ii) becomes:*  
*(ii') the restricted homomorphism  $\theta : FC(Q) \rightarrow \text{Aut}(K)$  is non injective.*

*Proof.* If  $\pi(G) = \text{Inn}(K) \rtimes \theta(Q)$ , then one has that  $\theta(Q) \cap \text{Inn}(K) = \{1\}$  so that  $\pi_K^{-1}(\theta(FC(Q))) = \{1\}$ , and with Theorem 3.2 condition (ii) becomes (ii').  $\square$

3.4.2. In case  $FC(Q)$  is finitely generated.

In case  $FC(Q)$  is finitely generated condition (ii) of Theorem 3.4 takes a better form; the reason is simply that under this hypothesis  $Z_Q(FC(Q))$  has finite index in  $Q$  which allows to rephrase the condition by mean of an homomorphism going from  $\ker \Phi$  to  $H^1(Z_Q(FC(Q)), Z(K))$ . For that matter the main proposition (3.7) below remains valid more generally whenever  $Z_Q(FC(Q))$  has a finite index in  $Q$ .

**Proposition-Definition 3.2.** *There exists an homomorphism*

$$\Xi : \ker \Phi \longrightarrow H^1(Z_Q(FC(Q)), Z(K)) ,$$

*defined by: given  $q \in \ker \Phi$ ,  $\Xi(q)$  is represented by the 1-cocycle  $d_q$ :*

$$\begin{array}{ccc} d_q : & Z_Q(FC(Q)) & \longrightarrow & Z(K) \\ & u & \longrightarrow & d_q(u) = [k^{-1}, u] \end{array}$$

*where  $\theta_q(x) = k x k^{-1}$  for all  $x \in K$ .*

*Proof.* The proof of proposition-definition 3.1 remains valid since for any  $q \in FC(Q)$ ,  $Z_Q(FC(Q)) \subset Z_Q(q)$ , so that the map  $\Xi$  is well defined. It remains to show that  $\Xi$  is an homomorphism. Let  $q_1, q_2 \in \ker \Phi$ , so that there exists  $k_1, k_2 \in K$  such that

$\forall x \in K$ ,  $\theta_{q_1}(x) = k_1 x k_1^{-1}$  and  $\theta_{q_2}(x) = k_2 x k_2^{-1}$ . Then  $q_1 q_2 \in FC(Q)$  and  $\theta_{q_1 q_2}(x) = k_1 k_2 x (k_1 k_2)^{-1}$ . Given  $u$  in  $Z_Q(FC(Q))$ ,  $u \in Z_Q(q_1) \cap Z_Q(q_2)$  so that:

$$\begin{aligned} d_{q_1 q_2}(u) &= [(k_1 k_2)^{-1}, u] \\ &= k_2^{-1} k_1^{-1} u k_1 k_2 u^{-1} \\ &= k_2^{-1} k_1^{-1} u k_1 u^{-1} u k_2 u^{-1} \\ &= k_2^{-1} d_{q_1}(u) u k_2 u^{-1} \\ &= d_{q_1}(u) k_2^{-1} u k_2 u^{-1} \\ &= d_{q_1}(u) d_{q_2}(u) \end{aligned}$$

and in particular  $[q_1 q_2] = [q_1][q_2]$  which proves that  $\Xi$  is an homomorphism.  $\square$

**Proposition 3.7** (in case  $FC(Q)$  is f.g.). *In the assumption of Theorems 3.1 to 3.4 if one moreover suppose that  $FC(Q)$  is finitely generated, condition (ii) becomes:*

(ii') *The homomorphism  $\Xi : \ker \Phi \longrightarrow H^1(Z_Q(FC(Q)), Z(K))$  is non injective.*

*Proof.* The proof of Theorem 3.4 remains valid in such case by noting that under this hypothesis for any  $q \in FC(Q)$ ,  $Z_Q(q)$  contains  $Z_Q(FC(Q))$  as a finite index subgroup. We leave the details as an exercise for the reader.  $\square$

*Remarks.*

– Under the more general hypothesis that  $Z_Q(FC(Q))$  has a finite index in  $Q$ , the conclusion remains valid; in particular when  $Q$  is abelian.

– Anytime  $Q$  is finitely generated and there is a uniform bound on the cardinal of each  $Q$ -conjugacy class in  $FC(Q)$ , the result applies. For a Theorem of M.Hall (cf. Theorem 4.7 in [LS]) asserts that each f.g. group contains only a finite number of subgroup of index a given  $n \in \mathbb{N}$ , and it follows that  $Z_Q(FC(Q))$  has finite index in  $G$  since it's an intersection of a finite number of finite index subgroups.

– By the same argument as in proposition 3.7, condition (ii) of Theorem 3.4 can be changed to:

(ii)  *$\ker \Phi$  contains a non-trivial finitely generated subgroup  $Q_0$  such that the homomorphism from  $Q_0$  to  $H^1(Z_Q(Q_0), Z(K))$  is non injective.*

– The conclusion of proposition 3.7 does not remain in general true when  $Z_Q(FC(Q))$  has an infinite index in  $Q$  (see example below); nevertheless condition (ii') turns to be a necessary condition to condition (ii).

*Example.* Here is an example which shows that when  $Z_Q(FC(Q))$  has infinite index in  $Q$  the conclusion of proposition 3.7 does not remain true. Consider the infinitely generated abelian group  $A = \mathbb{Z}^\omega \approx \langle a_1, \dots, a_n, \dots \mid [a_i, a_j], \forall i, j \in \mathbb{N}^* \rangle$  and  $Q = \langle A, t \mid \forall n \in \mathbb{N}^*, t^n a_n t^{-n} = a_n \rangle$ . Then for all  $n \in \mathbb{N}^*$ ,  $Z_Q(a_n)$  is the subgroup of  $Q$  generated by  $A$  and  $t^n$ ; it has a finite index in  $Q$ , so that  $A \subset FC(Q)$ , while  $Z_Q(FC(Q)) \subset Z_Q(A) = \bigcap_{n \in \mathbb{N}^*} Z_Q(a_n) = A$  has infinite index in  $Q$ .

Now set :  $K \approx F_2 \times \bigoplus_{n \in \mathbb{N}^*} (\mathbb{Z} \oplus \mathbb{Z})$  given by the presentation with countable sets of generators:  $\alpha, \beta, \alpha_1, \beta_1, \dots, \alpha_n, \beta_n, \dots$ , and relators:  $\forall i, j \in \mathbb{N}^*, [\alpha_i, \beta_j] = [\alpha, \alpha_i] = [\alpha, \beta_j] = [\beta, \alpha_i] = [\beta, \beta_j] = 1$ . Denote for all  $n \in \mathbb{N}^*$  by  $A_n$  the subgroup of  $K$  generated by  $\alpha_n, \beta_n$  naturally identified with  $\mathbb{Z} \oplus \mathbb{Z}$ , and consider an Anosov automorphism  $\varphi \in GL(2, \mathbb{Z})$  of  $\mathbb{Z} \oplus \mathbb{Z}$ . Define the homomorphism  $\theta : A \longrightarrow \text{Aut}(K)$  by  $\forall x \in K$ ,  $\theta(a_1)(x) = {}^\alpha x$  and  $\forall n \in \mathbb{N}^*$ ,  $\theta(a_{n+1})(x) = \varphi(x)$  if  $x \in A_n$  and  $\theta(a_{n+1})(x) = x$  otherwise. Let  $G_0 = K \rtimes_\theta A$ ; one has  $FC(K) = Z(K) = \bigoplus_{n \in \mathbb{N}^*} A_n$  and  $FC(K) \setminus 1$  contains only infinite  $\theta(A)$ -orbits despite  $G_0$  is not icc; indeed since  $\theta(a_1)(x) = {}^\alpha x$  and  $\theta(A)(\alpha) = \{\alpha\}$ ,  $\alpha^{-1} a_1$  has a finite conjugacy class in  $G_0$  and by proposition 3.7 the homomorphism  $\Xi : \ker \Phi \longrightarrow H^1(A, Z(K))$  is non injective: here  $\ker \Phi = \langle a_1 \rangle_A$  and  $\Xi(\ker \Phi) = \{0\}$  in  $H^1(A, Z(K))$ .

Now one extends  $\theta : A \longrightarrow \text{Aut}(K)$  to  $\theta' : Q \longrightarrow \text{Aut}(K)$  by setting  $\theta'(t)(\alpha) = \alpha\alpha_1$  and  $\theta'(t)(x) = x$  for all generator  $x$  of  $K$  distinct of  $\alpha$  (a direct calculation shows that  $\forall n \in \mathbb{N}^*$ ,  $\theta'(a_n) = \theta'(t^n)\theta'(a_n)\theta'(t^{-n})$ ). Define  $G = K \rtimes_{\theta'} Q$ ; for all  $n \in \mathbb{Z}^*$ ,  $\theta'(Q)(\alpha^n)$  is infinite so that  $G$  is not icc; Nevertheless, with all the above,  $\Xi : \ker \Phi \longrightarrow H^1(Z_Q(FC(Q)), Z(K))$  is non injective (here again  $\ker \Phi = \langle a_1 \rangle_Q$  and  $Z_Q(FC(Q)) = A$ ).

#### 4. EXTENSION OF GROUPS

Throughout this section we are concerned with a group  $G$  which decomposes as an extension:

$$\begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\rho} & Q \longrightarrow 1 \\ & & & & & & \downarrow \Theta \\ & & & & & & \text{Out}(K) \end{array}$$

The extension defines an homomorphism  $\Theta : Q \longrightarrow \text{Out}(K)$ . Consider also the homomorphism  $\pi : G \longrightarrow \text{Aut}(G)$  defined by  $\forall x \in K$ ,  $\pi(g)(x) = gxg^{-1}$ .

Let us denote in the following by  $Z := Z(K)$  the center of  $K$  and  $\forall q \in Q$ , by  $Z(q) := Z_Q(q)$  the centralizer of  $q$ .

**Proposition-Definition 4.1.** *Let  $\Theta : Q \longrightarrow \text{Out}(K)$  be the collective character associated to the extension. Any element  $q \in \ker \Theta$  defines an element  $[q]$  in the first cohomology group  $H^1(Z(q), Z)$ .*

This construction will be detailed in the next section.

**Theorem 4.1** (extension icc). *Let  $G$  decomposes as an extension, as above. Then  $G$  is not icc if and only if one of the following conditions is satisfied :*

- (i)  *$FC(K)$  contains a subgroup  $N \neq \{1\}$  normal in  $K$  preserved under the action of  $\pi(G)$  and such that either  $N$  is finite, or  $N \approx \mathbb{Z}^n$  and the image of  $\Theta(Q)$  in  $GL(n, \mathbb{Z})$  is finite.*
- (ii)  *$\Phi : FC(Q) \longrightarrow \text{Out}(K)$  is non injective, and  $\ker \Phi$  contains an element  $q \neq 1$  such that  $[q] = 0$  in  $H^1(Z(q), Z)$ .*

See §5.9 for a reformulation of (ii) as well as for an important particular case.

**4.1. Proof of the results.** This section is devoted to the proofs of the results enonced. We first define how an element  $q \in \ker \Theta$  defines  $[q]$  in the first cohomology group  $H^1(Z(q), Z)$ , which proves proposition-definition 4.1, and then we turn to the proof of Theorem 4.1.

**4.1.1. Proof of proposition-definition 4.1.** We fix a section  $s : Q \longrightarrow G$ , i.e. a map such that  $\forall q \in Q$ ,  $\rho \circ s(q) = q$ , and will rather denote  $\bar{q} := s(q)$ . The section  $s$  defines  $\forall q \in Q$  a lift  $\theta_q$  of  $\Theta(q)$  in  $\text{Aut}(K)$  defined by  $\theta_q(x) = \bar{q}x\bar{q}^{-1} = \bar{q}x$  for all  $x \in K$  (for more convenience we shall use both notations  $\theta_q(x)$  and  $\bar{q}x$ ). We will also write  $\theta_a(x) = \pi(a)(x) = {}^ax$  for any  $a, x \in K$ .

In the following we consider an element  $q \in \ker \Theta$ , that is  $q$  lies in  $Q$  and there exists  $k \in K$  such that  $\theta_q(x) = kxk^{-1}$  for all  $x \in K$ . Let  $u \in Z(q)$ ,  $uqu^{-1} = q$  in  $Q$ ; hence there exists an element  $\delta_q(u)$  in  $K$  defined by:

$$\bar{u}\bar{q}\bar{u}^{-1} = \bar{q}\delta_q(u) \quad \text{in } G.$$

**Proposition 4.1.** *For all  $u \in Z(q)$ ,  $\delta_q(u)^{-1}k^{-1}\bar{u}k$  lies in the center  $Z$  of  $K$ .*

*Proof.* Since  $u \in Z(q)$ ,  $\bar{u}\bar{q}\bar{u}^{-1} = \bar{q}\delta_q(u)$  one has  $\theta_u \circ \theta_q \circ \theta_u^{-1} = \theta_q \circ \theta_{\delta_q(u)}$  in  $\text{Aut}(K)$ . Hence  $\forall x \in K$ ,

$$\begin{aligned} & \theta_u \circ \theta_q \circ \theta_u^{-1}(x) = \theta_q \circ \theta_{\delta_q(u)}(x) \\ \iff & \theta_u(k \theta_u^{-1}(x) k^{-1}) = k \delta_q(u) x \delta_q(u)^{-1} k^{-1} \\ \iff & \theta_u(k) x \theta_u(k)^{-1} = k \delta_q(u) x \delta_q(u)^{-1} k^{-1} \end{aligned}$$

and it follows that  $\delta_q(u)^{-1} k^{-1} \bar{u} k = \delta_q(u)^{-1} k^{-1} \theta_u(k)$  commutes with  $x$  for all  $x \in K$ .  $\square$

**Definition.** Given  $q \in \ker \Theta$  and  $k \in K$  as above, we define the map:

$$\begin{aligned} d_q : Z(q) &\longrightarrow Z \\ u &\longrightarrow d_q(u) := \delta_q(u)^{-1} k^{-1} \bar{u} k. \end{aligned}$$

Note that in case of a semi-direct product (*i.e.* when the section  $s$  is an homomorphism) one recovers the map defined in proposition-definition 3.1. Note also that  $d_q$  *a priori* does depend both on the section  $s$  and on the choice of an element  $k \in K$  with  $\theta_q(x) = {}^k x$ .

**Proposition 4.2.** *The map  $d_q : Z(q) \longrightarrow Z$  is a 1-cocycle.*

*Proof.* We need to show that  $\forall u, v \in Z(q)$ ,  $d_q(uv) = d_q(u) \bar{u} d_q(v)$ . We denote by  $f : Q \times Q \longrightarrow K$  the 2-cocycle defined by the extension and by the section  $s$ , such that:

$$\forall x, y \in Q, \quad \bar{x}\bar{y} = \overline{xy} f(x, y).$$

Let  $u, v \in Z(q)$ . In order to compute  $d_q(uv)$  we need to compute first  $\delta_q(uv)$  and  $\theta_{uv}$ . Since  $\bar{uv} = \bar{u}\bar{v} f(u, v)$ , one has  $\theta_{uv} = \theta_u \circ \theta_v \circ \theta_{f(u,v)}^{-1}$ .

*Computation of  $\delta_q(uv)$ .* One has  $\bar{uv}\bar{q}\bar{uv}^{-1} = \bar{q}\delta_q(uv)$ .

$$\begin{aligned} \bar{uv}\bar{q}\bar{uv}^{-1} &= \bar{u}\bar{v} f(u, v)^{-1} \bar{q} f(u, v) \bar{v}^{-1} \bar{u}^{-1} \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{v} \bar{q} \bar{v}^{-1} \bar{v} f(u, v) \bar{u}^{-1} \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{u} \bar{q} \delta_q(v) \bar{u}^{-1} \bar{u} \bar{v} f(u, v) \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{u} \bar{q} \bar{u}^{-1} \bar{u} \delta_q(v) \bar{u} \bar{v} f(u, v) \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{q} \delta_q(u) \bar{u} \delta_q(v) \bar{u} \bar{v} f(u, v) \\ &= \bar{q} \bar{q}^{-1} \bar{u} \bar{v} f(u, v)^{-1} \delta_q(u) \bar{u} \delta_q(v) \bar{u} \bar{v} f(u, v) \end{aligned}$$

Hence  $\delta_q(uv) = \bar{q}^{-1} \bar{u} \bar{v} f(u, v)^{-1} \delta_q(u) \bar{u} \delta_q(v) \bar{u} \bar{v} f(u, v)$ .

By applying the definition, we obtain on the one hand:

$$\begin{aligned} d_q(uv) &= \delta_q(uv)^{-1} k^{-1} \theta_{uv}(k) \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{u} \delta_q(v)^{-1} \delta_q(u)^{-1} \bar{q}^{-1} \bar{u} \bar{v} f(u, v) k^{-1} \bar{u} \bar{v} f(u, v)^{-1} k f(u, v) \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{u} \delta_q(v)^{-1} \delta_q(u)^{-1} \bar{q}^{-1} \bar{u} \bar{v} f(u, v) \bar{q}^{-1} \bar{u} \bar{v} f(u, v)^{-1} k^{-1} \bar{u} \bar{v} (k f(u, v)) \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{u} \delta_q(v)^{-1} \delta_q(u)^{-1} k^{-1} \bar{u} \bar{v} (k f(u, v)) \\ &= \bar{u} \bar{v} f(u, v)^{-1} \bar{u} \delta_q(v)^{-1} \delta_q(u)^{-1} k^{-1} \bar{u} \bar{v} k \bar{u} \bar{v} f(u, v) \end{aligned}$$

and since  $d_q(uv)$  lies in the center  $Z$  of  $K$  (proposition 4.1),  $f(u, v)$ ,  $\delta_q(v)$  both lie in  $K$ , and  $K$  is preserved both by  $\theta_u$ ,  $\theta_v$ :

$$\begin{aligned} d_q(uv) &= \bar{u} \delta_q(v)^{-1} \delta_q(u)^{-1} k^{-1} \bar{u} \bar{v} k \\ &= \delta_q(u)^{-1} k^{-1} \bar{u} \bar{v} k \bar{u} \delta_q(v)^{-1}. \end{aligned}$$

On the other hand, since  $\delta_q(v) \in K$ ,  $d_q(v) \in Z$  and  $Z$  is a characteristic subgroup of  $K$ :

$$\begin{aligned} \bar{u} d_q(v) &= \bar{u} (\delta_q(v)^{-1} k^{-1} \bar{v} k) \\ &= \bar{u} (k^{-1} \bar{v} k \delta_q(v)^{-1}) \end{aligned}$$

so that:

$$\begin{aligned} d_q(u) \bar{u} d_q(v) &= \delta_q(u)^{-1} k^{-1} \bar{u} k \bar{u} k^{-1} \bar{u} \bar{v} k \bar{u} \delta_q(v)^{-1} \\ &= \delta_q(u)^{-1} k^{-1} \bar{u} \bar{v} k \bar{u} \delta_q(v)^{-1} \\ &= d_q(uv) \end{aligned}$$



which proves that  $d_q : Z(k) \rightarrow Z$  is a 1-cocycle.  $\square$

**Proposition 4.3.** *The cocycle  $d_q : Z(q) \rightarrow Z$  defines an element  $[q]$  in  $H^1(Z(q), Z)$  which only depends on  $q \in \ker \Theta$  and on the equivalence class of the extension  $G$  of  $K$  by  $Q$ .*

*Proof.* The cocycle  $d_q$  defines an element of  $H^1(Z(q), Z)$ . We need to prove that this element depends neither on  $k \in kZ$  nor on the section  $s$ .

Let  $k_1 \in kZ$ , say  $k_1 = kz$  for some  $z \in Z$ , which, as above, defines the 1-cocycle  $d'_q : Z(q) \rightarrow Z$  by:

$$\begin{aligned} d'_q(u) &= \delta_q(u)^{-1} k_1^{-1} \bar{u} k_1 \\ &= \delta_q(u)^{-1} z^{-1} k^{-1} \bar{u} k \bar{u} z \\ &= \delta_q(u)^{-1} k^{-1} \bar{u} k z^{-1} \bar{u} z \\ &= d_q(u) z^{-1} \bar{u} z \end{aligned}$$

$d_q, d'_q$  differ by the 1-coboundary :  $z \rightarrow z^{-1} \bar{u} z$ , so that they define the same element of  $H^1(Z(q), Z)$ .

Now we show that  $[q]$  does not depend on the section  $s$ . We proceed in two steps. First we change  $s(q) = \bar{q}$  into  $s(q) = \tilde{q} = \bar{q} \hat{q}$  for some  $\hat{q} \in K$ . Consider  $k' = k \hat{q}$ ; since for all  $x \in K$   $\tilde{q}x = k'x$  it allows to define a 1-cocycle that we denote by  $d'_q$  and whose class in  $H^1(Z(q), Z)$  does not depend on this choice. Obviously,  $d'_q(q) = d_q(q)$ . Given  $u \in Q$ ,  $u \neq q$ ,  $\delta_q(u)$  is changed into  $\delta'_q(u) = \hat{q}^{-1} \delta_q(u) \theta_u(\hat{q})$ , for:

$$\bar{u} \tilde{q} \bar{u}^{-1} = \bar{u} \bar{q} \hat{q} \bar{u}^{-1} = \bar{u} \bar{q} \bar{u}^{-1} \bar{u} \hat{q} = \bar{q} \delta_q(u) \bar{u} \hat{q} = \tilde{q} \hat{q}^{-1} \delta_q(u) \bar{u} \hat{q}$$

Hence:

$$d'_q(u) = \delta'_q(u)^{-1} k'^{-1} \theta_u(k') = \bar{u} \hat{q}^{-1} \delta_q(u)^{-1} \hat{q} \hat{q}^{-1} k^{-1} \bar{u} k \bar{u} \hat{q} = \bar{u} \hat{q}^{-1} \delta_q(u)^{-1} k^{-1} \bar{u} k \bar{u} \hat{q}$$

and since  $d'_q(u) \in Z$  and  $\bar{u} \hat{q} \in K$ , one obtains  $d'_q(u) = \delta_q(u)^{-1} k^{-1} \bar{u} k = d_q(u)$ . This ends the first step.

Now we change for  $u \in Q$ ,  $u \neq q$ ,  $s(u) = \bar{u}$  into  $s(u) = \tilde{u} = \bar{u} \hat{u}$  for some  $\hat{u} \in K$ . It changes  $\delta_q(u)$  into  $\delta'_q(u) = \delta_q(u) \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1})$  for:

$$\tilde{u} \bar{q} \tilde{u}^{-1} = \bar{u} \hat{u} \bar{q} \hat{u}^{-1} \bar{u}^{-1} = \bar{u} \bar{q} \bar{q}^{-1} \hat{u} \hat{u}^{-1} \bar{u}^{-1} = \bar{u} \bar{q} \bar{u}^{-1} \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}) = \bar{q} \delta_q(u) \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}).$$

Moreover,  $\tilde{u} k = \bar{u} k \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1})$  for:

$$\tilde{u} k = \bar{u} \hat{u} k \hat{u}^{-1} \bar{u}^{-1} = \bar{u} k \bar{u}^{-1} \bar{u} \hat{u}^{-1} \bar{u}^{-1} = \bar{u} k \bar{u}^{-1} \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}) = \bar{u} k \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1})$$

and we obtain:

$$\begin{aligned} d'_q(u) &= \delta'_q(u)^{-1} k^{-1} \tilde{u} k \\ &= (\bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}))^{-1} \delta_q(u)^{-1} k^{-1} \bar{u} k \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}) \\ &= \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1})^{-1} d_q(u) \bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}) \\ &= d_q(u) \end{aligned}$$

since  $d_q(u) \in Z$  and  $\bar{u} (\bar{q}^{-1} \hat{u} \hat{u}^{-1}) \in K$ . This ends the second step to show that  $[q]$  does not depend on the section  $s : Q \rightarrow G$ .  $\square$

4.1.2. *Proof of Theorem 4.1.* We now turn to the proof of Theorem 4.1.

*Step 1. Condition (i)  $\implies G$  is not icc;*

*Step 2.  $K$  contains  $u \neq 1$  with  ${}^G u$  finite  $\implies$  condition (i).*

In both cases the arguments of the facts 1 and 3 in the proof of Theorem 3.1 remain valid up to changing  $\theta(Q)$  by  $\pi(G \setminus K)$  and to noting that when  $N = \mathbb{Z}^n$ , both  $\Theta(Q)$  and  $\pi(G)$  have a natural homomorphic image in  $GL(n, \mathbb{Z})$  which do coincide.

*Step 3. Condition (ii)  $\implies G$  is not icc.*

Let  $q \in \ker \Phi$  such that  $q \neq 1$  and  $[q] = 0$  in  $H^1(Z_Q(q), Z(K))$ . We fix a section  $s : Q \rightarrow G$  and use the same notations as in §4.1.1. Let  $k \in K$  such that  $\theta_q(x) = k x k^{-1}$  for all  $x \in K$ . Since  $[q] = 0$ , there exists  $z \in Z$  such that for any  $u \in Z_Q(q)$ ,

$$d_q(u) := \delta_q(u)^{-1} k^{-1} \bar{u} k = z^{-1} \bar{u} z .$$

Let  $\omega = \bar{q} z k^{-1} \in G$ ; obviously  $\omega \neq 1$  and  $Z_G(\omega)$  contains  $K$ . For any  $u \in Z_Q(q)$ ,

$$\begin{aligned} \bar{u} \omega \bar{u}^{-1} &= \bar{u} \bar{q} z k^{-1} \bar{u}^{-1} \\ &= \bar{u} \bar{q} \bar{u}^{-1} \bar{u} z \bar{u} k^{-1} \\ &= \bar{q} \delta_q(u) \bar{u} z \bar{u} k^{-1} \\ &= \bar{q} k^{-1} \bar{u} k \bar{u} z^{-1} z \bar{u} z \bar{u} k^{-1} \\ &= \bar{q} k^{-1} \bar{u} k \bar{u} z^{-1} \bar{u} z \bar{u} k^{-1} z \\ &= \bar{q} k^{-1} z \\ &= \bar{q} z k^{-1} \\ &= \omega \end{aligned}$$

hence  $Z_G(\omega)$  contains the sub-extension of  $K$  by  $Z_Q(q)$ ; since  $q \in FC(Q)$ ,  $Z_Q(q)$  has finite index in  $Q$  so that  $Z_G(\omega)$  has finite index in  $G$ :  $G$  is not icc.

*Step 4.  $G$  not icc  $\implies$  condition (i) or (ii).*

Suppose  $G$  is not icc: there exists  $u \neq 1$  in  $G$  with  ${}^G u$  finite. According to step 2 if  $u \in K$ , condition (i) is satisfied. So suppose that  $u = \bar{q} k^{-1}$  for some  $k \in K$  and  $q \neq 1$  in  $Q$ ;  $q$  necessarily lies in  $FC(Q)$ . Let  $K_0 = Z_G(u) \cap K$ ; it has a finite index in  $K$ , and  $\forall x \in K_0$ ,  $\theta_q(x) = k x k^{-1}$ . If  $K_0 \neq K$ , let  $h \in K \setminus K_0$  and  $\omega = [u, h] \neq 1$ ; then  $\omega$  lies in  $K$  and  ${}^G \omega$  is finite since  $Z_G(\omega) \supset Z_G(u) \cap h Z_G(u) h^{-1}$  has finite index in  $G$ , so that, with the step 2, condition (i) is satisfied.

Suppose in the following that  $K = K_0$ , i.e.  $\theta_q$  is inner,  $\forall x \in K$ ,  $\theta_q(x) = k x k^{-1}$ . Let  $Q_0 = \rho(Z_G(u))$ ;  $Q_0$  is included in  $Z_Q(q)$ . If  $Z_Q(q) \setminus Q_0 \neq \emptyset$ , let  $p \in Z_Q(q) \setminus Q_0$ ; one has  $w = [\bar{p}, u] \neq 1$  in  $G$ . Let's prove that  $w$  lies in  $Z(K)$ :

$$\begin{aligned} [\bar{p}, u] &= \bar{p} \bar{q} k^{-1} \bar{p}^{-1} k \bar{q}^{-1} \\ &= \bar{p} \bar{q} \bar{p}^{-1} \bar{p} k^{-1} k \bar{q}^{-1} \\ &= \bar{q} \delta_q(p) \bar{p} k^{-1} k \bar{q}^{-1} \\ &= k \delta_q(p) \bar{p} k^{-1} k k^{-1} \\ &= k \delta_q(p) \bar{p} k^{-1} \end{aligned}$$

is conjugated in  $K$  to  $d_q(p)^{-1} \in Z(K)$ ; hence  $w = d_q(p)^{-1} \in Z(K) \subset K$ . Furthermore  $w$  has a finite conjugacy class in  $G$  since  $Z_G(w)$  contains  $Z_G(u) \cap \bar{p} Z_G(u) \bar{p}^{-1}$  which has a finite index in  $G$ . According to step 2, condition (i) is satisfied.

Suppose now that  $\rho(Z_G(u)) = Z_Q(q)$ ; let  $v \in Z_Q(q)$ , since  $Z_G(u) \supset K$  one has  $\bar{v} u \bar{v}^{-1} = u$  in  $G$ . Then:

$$\begin{aligned} \bar{q} k^{-1} &= \bar{v} \bar{q} k^{-1} \bar{v}^{-1} \\ &= \bar{v} \bar{q} \bar{v}^{-1} \bar{v} k^{-1} \\ &= \bar{q} \delta_q(v) \bar{v} k^{-1} \\ \implies \delta_q(v)^{-1} k^{-1} \bar{v} k &= d_q(v) = 1 . \end{aligned}$$

Hence for all  $v \in Z_Q(q)$ ,  $d_q(v) = 1$ , so that  $[q] = 0$  in  $H^1(Z_Q(q), Z(K))$ . Condition (ii) is satisfied.  $\square$

## 5. IMPROVED RESULTS FOR PARTICULAR CASES OF EXTENSIONS

We now consider particular cases. Note that all particular cases in §5.1 to §5.10 remains valid in case of a split extension and that in such case condition (i) of Theorem 4.1 rephrases as condition (i) of Theorem 3.1.

### 5.1. In case the kernel is centerless or simple.

**Proposition 5.1** (extension of centerless group icc). *Let  $G$  be an extension:*

$$1 \longrightarrow K \text{ centerless} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*Then  $G$  is not icc if and only if either:*

- (i) *condition (i) of Theorem 4.1 is satisfied, or*
- (ii)  *$\Phi : FC(Q) \longrightarrow Out(K)$  is non-injective.*

*Proof.* The proof is a straightforward consequence of Theorem 4.1 since in that case for any  $q \in \ker \Phi$ ,  $H^1(Z_Q(q), Z(K)) = \{0\}$ .  $\square$

In particular, since simple groups with non trivial center are exactly  $\mathbb{Z}/p\mathbb{Z}$  for some  $p$  prime.

**Proposition 5.2** (extension of simple group icc). *Let  $G$  be an extension:*

$$1 \longrightarrow K \text{ simple} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*Then  $G$  is icc if and only if both:*

- (i)  *$K$  is infinite,*
- (ii)  *$\Phi : FC(Q) \longrightarrow Out(K)$  is injective.*

*Proof.* It's a straightforward application of proposition 5.1 since a simple group is icc if and only if it is infinite (property 5), and has a trivial center, except for  $\mathbb{Z}/p\mathbb{Z}$  with  $p$  prime.  $\square$

### 5.2. In case of a centerless by simple group.

**Proposition 5.3** (centerless by simple). *Let  $G$  be defined by an extension:*

$$1 \longrightarrow K \text{ centerless} \longrightarrow G \longrightarrow Q \text{ simple} \longrightarrow 1$$

*then  $G$  is not icc if and only if either*

- (i) *condition (i) of Theorem 4.1 is satisfied, or*
- (ii) *the extension is equivalent to  $K \times Q$ .*

*Proof.* Clearly condition (i) of Theorem 4.1 and  $G \approx K \times Q$  both imply that  $G$  is not icc. If condition (ii) of Theorem 4.1 is satisfied, on the one hand, since  $Q$  is simple,  $\Theta(Q) = \{1\}$ , on the other since  $K$  has a trivial center there is up to equivalence a unique extension of  $G$  by  $Q$  with  $\Theta(Q) = \{1\}$  (cf. Corollary 6.8 [Br]), so that the extension is equivalent to  $K \times Q$ . The result follows from Theorem 4.1.  $\square$

Since icc groups are centerless, one obtains:

**Corollary 5.1** (icc by simple groups icc). *Let  $G$  be defined by an extension:*

$$1 \longrightarrow K \text{ icc} \longrightarrow G \longrightarrow Q \text{ simple} \longrightarrow 1$$

*then  $G$  is icc if and only if the extension is not equivalent to  $K \times Q$ .*

### 5.3. In case the kernel is abelian.

**Proposition 5.4** (extension of abelian group icc). *Let  $G$  be an extension:*

$$1 \longrightarrow K \text{ abelian} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*and  $\theta : Q \longrightarrow \text{Aut}(K)$  the associated homomorphism.*

*Then  $G$  is icc if and only if both:*

- (i)  $K \setminus \{1\}$  contains only infinite  $\theta(Q)$ -orbits, and
- (ii) the restricted homomorphism  $\theta : FC(Q) \longrightarrow \text{Aut}(K)$  is injective.

*Proof.* Since  $\text{Inn}(K) = \{1\}$  the homomorphism  $\Theta : Q \longrightarrow \text{Out}(K)$  defines an homomorphism  $\theta : Q \longrightarrow \text{Aut}(K)$ . Given  $u \in K$ ,  $G_u$  coincides with the  $\theta(Q)$ -orbit of  $u$ . Clearly if either (i) or (ii) is not satisfied then  $G$  is not icc. Conversely suppose  $G$  is not icc, and let  $u \neq 1$  with  $G_u$  finite. If  $u \in K$  then (i) is not satisfied. If  $u \in G \setminus K$  then let  $q = \rho(u)$ ; necessarily  $q \in FC(Q) \setminus \{1\}$ . Let  $K_0 = K \cap Z_G(u)$ , it has a finite index in  $K$  and  $\theta(q)$  restricts to the identity on  $K_0$ . If  $K_0 \neq K$ , let  $v \in K \setminus K_0$  and  $w = [v, u]$ . Then  $w \in K$ ,  $w \neq 1$  and  $G_w$  is finite since its centralizer contains  $Z_G(u) \cap v Z_G(u) v^{-1}$  so that it has a finite index in  $G$ . Hence either (i) is not satisfied, or  $\theta(q)$  is the identity and (ii) is not satisfied.  $\square$

*Example.* Consider a metabelian group  $G$ , i.e. whose derived group  $[G, G]$  is abelian. Then  $G$  is icc if and only if  $\theta : G/[G, G] \longrightarrow \text{Aut}([G, G])$  is injective and  $[G, G] \setminus \{1\}$  contains only infinite  $\theta(G/[G, G])$ -orbits, if and only if  $[G, G] \setminus \{1\}$  contains only infinite  $\theta(G/[G, G])$ -orbits and  $[G, G]$  is a maximal abelian subgroup (see also example in §5.10).

### 5.4. In case the quotient is infinite cyclic.

**Proposition 5.5** (extension by infinite cyclic). *Let  $G$  be a group extension:*

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1$$

*Then  $G$  is not icc if and only if either:*

- (i) condition (i) of Theorem 4.1 is satisfied, or
- (ii)  $\Theta : \mathbb{Z} \longrightarrow \text{Out}(K)$  is non injective.

*Proof.* If (i) is satisfied then according to Theorem 4.1.(i),  $G$  is not icc. Suppose now that  $\Theta$  is non injective; let  $t$  be a generator of  $\mathbb{Z}$ , then there exists  $n \in \mathbb{Z}$  and  $k \in K$  such that  $\forall x \in K, \bar{t}^n x \bar{t}^{-n} = k x k^{-1}$ . Hence  $Z_G(k^{-1} \bar{t}^n)$  contains  $K$  and  $\bar{t}^n$  and then has a finite index in  $G$ ;  $G$  is not icc. Conversely, if  $G$  is not icc, Theorem 4.1 shows that either condition (i) or (ii) is satisfied.  $\square$

### 5.5. In case the kernel is icc.

**Proposition 5.6** (extension of icc group icc). *Let  $G$  be an extension:*

$$1 \longrightarrow K_{icc} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*then  $G$  is icc if and only if  $\Phi : FC(Q) \longrightarrow Out(K)$  is injective.*

*Proof.* Since  $K$  is icc both  $FC(K)$  and  $Z(K)$  are trivial and conclusion follows from proposition 5.1.  $\square$

Obviously one obtains:

**Corollary 5.2.** *The icc property is stable by extension.*

### 5.6. In case the kernel is free.

**Proposition 5.7** (extension of free group icc). *Let  $G$  be an extension:*

$$1 \longrightarrow K_{free} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*Then  $G$  is icc if and only if both  $K \not\approx \mathbb{Z}$  and  $\Phi : FC(Q) \longrightarrow Out(K)$  is injective.*

*Proof.* A free group is either infinite cyclic or icc. If  $K \approx \mathbb{Z}$ , since  $Aut(\mathbb{Z})$  has order 2, condition (i) of Theorem 4.1 is satisfied, and otherwise the conclusion follows from proposition 5.6.  $\square$

### 5.7. In case of a finite extension.

**Proposition 5.8** (finite extension icc). *Let  $G$  be a finite extension:*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \text{ finite} \longrightarrow 1$$

*Then  $G$  is icc if and only if both  $K$  is icc and  $\Theta : Q \longrightarrow Out(K)$  is injective.*

*Proof.* On the one hand if  $G$  is icc then necessarily  $K$  is icc; for suppose on the contrary that  $\exists k \in K \setminus \{1\}$  such that  $Z_K(k)$  has a finite index in  $K$ ; since  $K$  has a finite index in  $G$  and  $Z_K(k) \subset Z_G(k)$ ,  $Z_G(k)$  has a finite index in  $G$ ,  ${}^G k$  is finite and  $G$  is not icc. On the other hand suppose that  $K$  is icc; since  $Q$  is finite,  $FC(Q) = Q$  so that with proposition 5.6,  $G$  is icc if and only if  $\Theta : Q \longrightarrow Out(K)$  is injective.  $\square$

Moreover when  $Q$  is a simple finite group one obtains:

**Proposition 5.9** (extension by finite simple). *Let  $G$  be an extension:*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \text{ finite simple} \longrightarrow 1$$

*Then  $G$  is icc if and only if  $K$  is icc and the extension is not equivalent to  $K \times Q$ .*

*Proof.* It follows from proposition 5.8 and corollary 5.1.  $\square$

*Example.* Let  $p$  be a prime; a group containing an icc normal subgroup  $K$  with index  $p$  is either icc or  $K \times \mathbb{Z}/p\mathbb{Z}$ .

### 5.8. In case the kernel is hyperbolic.

For definition and basic facts upon hyperbolic groups we refer the reader to [CDP].

**Proposition 5.10** (hyperbolic group icc). *Let  $G$  be a hyperbolic group; then  $G$  is icc if and only if  $G$  is non-elementary and does not contain a non-trivial finite characteristic (respectively normal) subgroup.*

*Proof.* If  $G$  is elementary, i.e. either finite or virtually  $\mathbb{Z}$ , or if  $G$  contains a non-trivial finite normal (in particular characteristic) subgroup, then clearly  $G$  is not icc. Conversely suppose that  $G$  is non elementary and not icc. Since in hyperbolic groups infinite order elements have a virtually cyclic centralizer (cf. Corollary 7.2, [CDP]),  $FC(G)$  is periodic. Since hyperbolic groups contain finitely many conjugacy classes of torsion element (cf. Lemma 3.5 in [CDP]),  $FC(G)$  is finite and the conclusion holds.  $\square$

**Proposition 5.11** (extension of hyperbolic group icc). *Let  $G$  be a group extension:*

$$1 \longrightarrow K \text{ hyperbolic} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*Then  $G$  is icc if and only if both:*

- (i)  $K$  is icc
- (ii)  $\Phi : FC(Q) \longrightarrow Out(K)$  is injective.

*Proof.* If condition (i) and (ii) are satisfied then  $G$  is icc, for, according to theorem 4.1 if  $G$  is not icc then either condition (i) or (ii) is not satisfied. Conversely suppose that  $G$  is icc. Necessarily  $K$  is non-elementary for otherwise  $K$  would be finite or would contain a characteristic infinite cyclic subgroup, and according to Theorem 4.1.(i) it would contradict that  $G$  is icc. Now a non-elementary hyperbolic has a finite center (which follows from Corollary 7.2, [CDP]). Necessarily  $Z(K) = \{1\}$  for otherwise with Theorem 4.1.(i) it would contradict that  $G$  is icc. Hence proposition 5.1 applies so that  $\Phi$  is injective and condition (i) of Theorem 4.1 is not satisfied. By applying proposition 5.10,  $K$  not icc implies condition (i) of Theorem 4.1; hence  $K$  is icc.  $\square$

### 5.9. In case $FC(Q)$ is finitely generated.

As in section 3.4.2, and for similar reasons, in case  $FC(Q)$  is finitely generated condition (ii) of Theorem 4.1 can be rephrased by mean of injectivity of an homomorphism from  $\ker \Phi$  to  $H^1(Z_Q(FC(Q)), Z(K))$ . This fact (proposition (5.12) below) remains valid more generally whenever  $Z_Q(FC(Q))$  has a finite index in  $Q$ .

**Proposition-Definition 5.1.** *There exists an homomorphism*

$$\Xi : \ker \Phi \longrightarrow H^1(Z_Q(FC(Q)), Z(K)) ,$$

defined by: given  $q \in \ker \Phi$ ,  $\Xi(q)$  is represented by the 1-cocycle  $d_q$ :

$$\begin{aligned} d_q : Z_Q(FC(Q)) &\longrightarrow Z(K) \\ u &\longrightarrow d_q(u) = \delta_q(u)^{-1} k^{-1} \bar{u} k \end{aligned}$$

where  $\theta_q(x) = k x k^{-1}$  for all  $x \in K$ .

*Proof.* The proof of proposition-definition 4.1 remains valid since for any  $q \in FC(Q)$ ,  $Z_Q(FC(Q)) \subset Z_Q(q)$ , so that the map  $\Xi$  is well defined. It remains to show that  $\Xi$  is an homomorphism. Let  $q_1, q_2 \in \ker \Phi$ , so that there exists  $k_1, k_2 \in K$  such that  $\forall x \in K$ ,  $\theta_{q_1}(x) = k_1 x k_1^{-1}$  and  $\theta_{q_2}(x) = k_2 x k_2^{-1}$ . Then  $q_1 q_2 \in FC(Q)$  and since  $\overline{q_1 q_2} = \overline{q_1} \overline{q_2}$   $f(q_1, q_2)$ :

$$\theta_{q_1 q_2}(x) = k_1 k_2 f(q_1, q_2)^{-1} x f(q_1, q_2) (k_1 k_2)^{-1}.$$

*Computation of  $\delta_{q_1 q_2}$ .* Given  $u \in Q$ :

$$\begin{aligned} \bar{u} \overline{q_1} \overline{q_2} \bar{u}^{-1} &= \overline{q_1} \delta_{q_1}(u) \overline{q_2} \delta_{q_2}(u) = \overline{q_1} \overline{q_2} k_2^{-1} \delta_{q_1}(u) \delta_{q_2}(u) \\ &= \bar{u} \overline{q_1 q_2} f(q_1, q_2) \bar{u}^{-1} = \overline{q_1 q_2} \delta_{q_1 q_2}(u) \bar{u} f(q_1, q_2) \\ \implies \delta_{q_1 q_2}(u) &= f(q_1, q_2) k_2^{-1} \delta_{q_1}(u) \delta_{q_2}(u) \bar{u} f(q_1, q_2)^{-1}. \end{aligned}$$

*Computation of  $d_{q_1 q_2}$ .* For  $u \in Q$ , by definition:

$$\begin{aligned} d_{q_1 q_2}(u) &= \bar{u} f(q_1, q_2) \delta_{q_2}(u)^{-1} k_2^{-1} \delta_{q_1}(u)^{-1} f(q_1, q_2)^{-1} f(q_1, q_2) k_2^{-1} k_1^{-1} \bar{u} k_1 \bar{u} k_2 \bar{u} f(q_1, q_2)^{-1} \\ &= \bar{u} f(q_1, q_2) \delta_{q_2}(u)^{-1} k_2^{-1} \delta_{q_1}(u)^{-1} k_2^{-1} k_1^{-1} \bar{u} k_1 \bar{u} k_2 \bar{u} f(q_1, q_2)^{-1} \\ &= \bar{u} f(q_1, q_2) \delta_{q_2}(u)^{-1} k_2^{-1} \delta_{q_1}(u)^{-1} k_1^{-1} \bar{u} k_1 \bar{u} k_2 \bar{u} f(q_1, q_2)^{-1} \\ &= \bar{u} f(q_1, q_2) \delta_{q_2}(u)^{-1} k_2^{-1} d_{q_1}(u) \bar{u} k_2 \bar{u} f(q_1, q_2)^{-1} \\ &= \bar{u} f(q_1, q_2) d_{q_1}(u) \delta_{q_2}(u)^{-1} k_2^{-1} \bar{u} k_2 \bar{u} f(q_1, q_2)^{-1} \\ &= \bar{u} f(q_1, q_2) d_{q_1}(u) d_{q_2}(u) \bar{u} f(q_1, q_2)^{-1} \\ &= d_{q_1}(u) d_{q_2}(u) \end{aligned}$$

(keep in mind that  $\forall q \in \ker \Phi$ ,  $\forall u \in Q$ ,  $d_q(u) \in Z(K)$ .)

Hence  $[q_1 q_2] = [q_1] [q_2]$  which proves that  $\Xi$  is an homomorphism.  $\square$

**Proposition 5.12** (In case  $FC(Q)$  is finitely generated). *In the assumption of Theorems 4.1 if one moreover suppose that  $FC(Q)$  is finitely generated, condition (ii) becomes:*

(ii') *The homomorphism  $\Xi : \ker \Phi \longrightarrow H^1(Z_Q(FC(Q)), Z(K))$  is non injective.*

*Proof.* The proof of Theorem 4.1 remains valid in such case if one changes  $Z_Q(q)$  into  $Z_Q(FC(Q))$  by noting that under the hypothesis for any  $q \in FC(Q)$ ,  $Z_Q(q)$  contains  $Z_Q(FC(Q))$  as a finite index subgroup.  $\square$

*Remarks.*

– As in §3.4.2 : the conclusion remains more generally true under the hypothesis that  $Z_Q(FC(Q))$  has a finite index in  $Q$ ; in particular when  $Q$  is finitely generated and all  $Q$ -conjugacy classes in  $FC(Q)$  have a uniformly bounded cardinal; and on the contrary, the conclusion does not hold in general when  $Z_Q(FC(Q))$  has an infinite index in  $Q$  (an example is given in §3.4.2).

– By the same argument as in the proof, condition (ii) of Theorem 4.1 can be changed to:

(ii)  *$\ker \Phi$  contains a non-trivial finitely generated subgroup  $Q_0$  such that the homomorphism from  $Q_0$  to  $H^1(Z_Q(Q_0), Z(K))$  is non injective.*

And moreover condition (ii) implies condition (ii').

### 5.10. In case of an extension by an abelian group.

By applying a result in the last section, one obtains:

**Proposition 5.13** (extension by abelian). *Let  $G$  be a group which decomposes as an extension by an abelian group:*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \text{ abelian} \longrightarrow 1$$

*then  $G$  is icc if and only if condition (i) of Theorem 4.1 fails and  $\Xi : \ker \Theta \longrightarrow H^1(Q, Z(K))$  is injective.*

*Proof.* Since  $Q$  is abelian one has  $FC(Q) = Q$  and  $Z_Q(FC(Q)) = Q$  so that Proposition 5.12 applies, and one concludes that  $G$  is icc if and only if condition (i) of Theorem 4.1 fails and  $\Xi : \ker \Theta \longrightarrow H^1(Q, Z(K))$  is injective.  $\square$

**Proposition 5.14.** *Under the same hypothesis,  $\Xi : \ker \Theta \longrightarrow H^1(Q, Z(K))$  is injective if and only if  $Z(G) \subset K$ .*

*Proof.* Let  $q \neq 1 \in \ker \Theta$  such that  $\Xi(q) = 0$  in  $H^1(Q, Z(K))$ , the same argument as in the step 3 in the proof of Theorem 4.1 shows that  $G \setminus K$  contains an element in  $Z(G)$ . Reciprocally let  $g \in G \setminus K$  with  $g \in Z(G)$ ; let  $\rho : G \longrightarrow Q$  and  $q = \rho(g)$  so that (with the same notation as in §4.1) there exists  $k \in K$  such that  $g = \bar{q}k^{-1}$ . Since  $g \in Z(G)$  one has  $\forall x \in K, \Theta(q)(x) = {}^kx$ . For any arbitrary element  $u \in Q$ , since  $g \in Z(G)$ , one has  $\bar{u}g\bar{u}^{-1} = g$ , hence with  $g = \bar{q}k^{-1}$ ,

$$\bar{u}\bar{q}k^{-1}\bar{u}^{-1} = \bar{u}\bar{q}\bar{u}^{-1}\bar{u}k^{-1} = \bar{q}k^{-1} \implies \bar{u}\bar{q}\bar{u} = \bar{q}k^{-1}\bar{u}k$$

so that  $\forall u \in Q, \delta_q(u) = k^{-1}\bar{u}k$  and:

$$d_q(u) = \delta_q(u)^{-1}k^{-1}\bar{u}k = \bar{u}k^{-1}k k^{-1}\bar{u}k = 1$$

hence  $[q] = 0$  in  $H^1(Q, Z(K))$  which shows that  $\Xi$  is non injective.  $\square$

*Example.* Let  $G$  be a *non-perfect* group, i.e. whose derived group  $[G, G]$  is a proper subgroup (in particular when  $G$  is *solvable*). If  $G$  is not abelian, it decomposes as an extension with kernel  $[G, G]$ ; let  $\Theta : G/[G, G] \longrightarrow \text{Out}([G, G])$  be the associated homomorphism. Then  $G$  is icc if and only if condition (i) of Theorem 4.1 fails and  $\Xi : \ker \Theta \longrightarrow H^1(G/[G, G], Z([G, G]))$  is injective, il and only if condition 4.1.(i) fails and  $G$  is centerless.

## 6. EXAMPLES

As an example of application we deduce easily a characterization of icc property both for HNN extensions of groups and amalgams of groups, recovering results proved in [Co]; these results answer two questions of Pierre de la Harpe appearing in [Ha], Problems 27, 28. We also strenghten proposition 5.8 on finite extensions and generalize to all groups containing a proper finite index subgroup.

**6.1. HNN extensions.** For basic facts upon HNN extensions of groups we refer the reader to [LS] (we only make use of their fundamental property: the Britton's Lemma). Let  $A$  be a non-trivial group with subgroups  $C, C'$  and let  $\varphi : C \longrightarrow C'$  be an isomorphism. Let  $G := A_{*\varphi}$  the HNN extension:  $G \approx \langle A, t \mid \forall c \in C, tct^{-1} = \varphi(c) \rangle$ .

Consider an epimorphism  $\rho : G \longrightarrow \mathbb{Z}$  such that  $A \subset \ker \rho$  and  $\rho(t)$  generates  $\mathbb{Z}$ , and let  $K := \ker \rho$  so that  $G$  decomposes as a split extension:

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\rho} \mathbb{Z} \longrightarrow 1$$



and let  $\theta : \mathbb{Z} \longrightarrow \text{Aut}(K)$  the associated homomorphism:  $\forall x \in K, \theta(x) = t x t^{-1}$ . It gives rise to the homomorphism  $\Theta : \mathbb{Z} \longrightarrow \text{Out}(K)$ .

If  $C = C' = A$  then  $\ker \rho = A$  and  $G$  is an extension of  $A$  by  $\mathbb{Z}$ . In such case, Proposition 5.4 applies to characterise icc property. So we suppose in the following, and without loss of generality, that  $C$  is a proper subgroup of  $A$ .

• *The homomorphism  $\Theta$  is injective.* Suppose on the contrary that there exists  $n \in \mathbb{N}$  and  $k \in K$  such that  $\forall x \in K, t^n x t^{-n} = k x k^{-1}$ . Let  $\alpha \in A \setminus C$ ;  $t^n \alpha t^{-n}$  is reduced so that  $k \notin A$ ,  $k$  has reduced form:

$$k = k_0 t^{\varepsilon_1} k_1 \cdots t^{\varepsilon_p} k_p \quad \text{with } k_0, k_i \in A, \varepsilon_i = \pm 1, \forall i = 1, \dots, p$$

for some  $p \geq n$ . On the one hand, since  $t^n k t^{-n} = k$ , necessarily  $\varepsilon_1 = -1$  or  $\varepsilon_p = 1$ ; on the other hand since  $t^{-n} k t^n = k$ , necessarily  $\varepsilon_1 = 1$  or  $\varepsilon_p = -1$ ; it follows that  $\varepsilon_1 = \varepsilon_p = \pm 1$ . If  $\varepsilon_1 = \varepsilon_p = 1$  then  $k \alpha k^{-1}$  is also reduced and  $t^n \alpha t^{-n} = k \alpha k^{-1}$  implies  $t^n \in kA$  which cannot occur since  $t^n \notin K$ . If  $\varepsilon_1 = \varepsilon_p = -1$ : similarly  $t^{-n} \alpha t^n = k^{-1} \alpha k$ ; if  $\alpha \in C'$  the right term is reduced while the left term is not which contradicts that  $p \geq n$ ; if  $\alpha \notin C'$  the same argument as before leads to a contradiction.

•  *$\theta(\mathbb{Z})$  has only infinite orbits in  $K \setminus C \cap C'$ .* Suppose on the contrary that there exists  $k \in K \setminus C \cap C'$  with a finite  $\theta(\mathbb{Z})$ -orbit. Then  $k \in K \setminus A$ , for suppose without loss of generality that  $k \in A \setminus C$ , then  $\{t^n k t^{-n}; n \in \mathbb{N}\}$  is infinite and contained in the  $\theta(Q)$ -orbit of  $k$ . Hence  $k$  has reduced form:

$$k = k_0 t^{\varepsilon_1} k_1 \cdots t^{\varepsilon_p} k_p \quad \text{with } k_0, k_i \in A, \varepsilon_i = \pm 1, \forall i = 1, \dots, p$$

for some  $p \geq 1$ . Necessarily  $\varepsilon_1 = \varepsilon_p$  for otherwise the  $\theta(\mathbb{Z})$ -orbit of  $k$  would contain the infinite set  $\{t^{\varepsilon_1 n} k t^{\varepsilon_p n}; n \in \mathbb{N}\}$ , so suppose without loss of generality that  $\varepsilon_1 = \varepsilon_p = -1$ . An immediate induction shows that indeed  $\forall i = 1, \dots, p, \varepsilon_i = -1$ . This leads to a contradiction since  $k \in K$  implies that  $\sum_{i=1}^p \varepsilon_i = 0$ .

By collecting the facts proved above and applying Theorem 3.1 and Proposition 5.4 one obtains:

**Proposition 6.1** (HNN extensions icc). *Let  $G = A_{*\varphi}$  be an HNN extension.*

• *If the HNN extension is non degenerate. Then  $G$  is not icc if and only if:*

(i)  *$FC(A) \cap C \cap C'$  contains a non-trivial subgroup  $N$ ,  $\theta(\mathbb{Z})$ -stable such that either  $N$  is finite or  $N \approx \mathbb{Z}^n$  and the image of  $\theta(\mathbb{Z})$  in  $GL(n, \mathbb{Z})$  is finite.*

• *If the HNN extension is degenerate. Then  $G$  is not icc if and only either (i) is satisfied or the natural homomorphism  $\Theta : \mathbb{Z} \longrightarrow \text{Out}(A)$  is non-injective.*

*Example.* Consider the Baumslag-Solitar group  $BS(m, n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$ . Then  $G$  is icc if and only if  $m \neq \pm n$ .

**6.2. Amalgamated product.** In this section let  $A, B$  be groups,  $C, C'$  proper subgroups respectively of  $A$  and  $B$ ,  $\varphi : C \longrightarrow C'$  an isomorphism and consider the amalgamated product  $G = A *_C B \approx \langle A, B \mid \forall c \in C, c = \varphi(c) \rangle$ . In the following we identify  $C$  with a subgroup both of  $A$  and  $B$ ; we refer the reader to [LS] for elementary facts upon amalgamated product (we only make use of their the fundamental property: the normal form Theorem).

Denote by:

$$\tilde{C}_A = \bigcap_{a \in A} a C a^{-1}, \quad \tilde{C}_B = \bigcap_{b \in B} b C b^{-1}, \quad \tilde{C} = \tilde{C}_A \cap \tilde{C}_B$$

the largest subgroups of  $C$  normal respectively in  $A, B$  and  $G$ . Denote by  $\pi : A *_C B \longrightarrow \text{Aut}(\tilde{C})$  the homomorphism defined by  $\forall g \in A *_C B, \forall x \in \tilde{C}, \pi(g)(x) = g x g^{-1}$ .

• *Every element of  $A \setminus \tilde{C}$  has an infinite  $G$ -conjugacy class.* Let  $\tilde{a} \in A \setminus C$  and  $b \in B \setminus C$ . Then the elements  $(ab)^n \tilde{a} (ab)^{-n}$ ,  $n \in \mathbb{N}$  are pairwise disjoint elements of  ${}^G \tilde{a}$ , so that  ${}^G \tilde{a}$  is infinite. Now let  $x \in A \setminus \tilde{C}$ , a conjugate of  $x$  lies in  $(A \cup B) \setminus C$  so that  ${}^G x$  is infinite.

• *If  $[A : C] > 2$  then every element of  $G \setminus (A \cup B)$  has an infinite conjugacy class.* Let  $u \in G \setminus (A \cup B)$ ; up to conjugacy by an element of  $A \cup B$  we suppose that  $u$  has reduced form  $u = a_1 b_1 \cdots a_n b_n$  for some  $n \in \mathbb{N}^*$  with  $\forall i = 1, \dots, n: a_i \in A \setminus C$  and  $b_i \in B \setminus C$ . Let  $b \in B \setminus C$  and  $A \in A \setminus C$  such that  $a$  and  $a_1^{-1}$  lie in different cosets of  $A/C$ . Then the elements  $(ba)^n u (ba)^{-n}$  for  $n \in \mathbb{N}$  are pairwise distinct elements in  ${}^G u$ , so that  ${}^G u$  is infinite.

We now distinguish two cases:

• *First case: the degenerate case,  $[A : C] = [B : C] = 2$ .* In such case  $C = \tilde{C}$  is normal in  $A, B$  and  $G$ . Then in case  $C = \{1\}$ ,  $G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$  is not icc and  $\Theta : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Out}(C)$  is non injective, and otherwise  $G$  decomposes as an extension:

$$1 \longrightarrow C \longrightarrow A *_C B \longrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

$FC(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$  consists in the characteristic infinite cyclic subgroup of the infinite dihedral group. Note that  $\Theta : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Out}(C)$  is injective if and only if its restriction to  $FC(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$  is injective. Note also that whenever  $\Theta$  is non injective, then  $G$  is not icc: it follows as in the proof of proposition 5.5 since any non trivial element of  $FC(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$  generates a finite index subgroup of  $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ . The result follows readily from Theorem 4.1.

• *Second case: the non-degenerate case.* If  $\tilde{C} = \{1\}$  the two facts proved above show that  $G$  is icc. Otherwise, denote  $\bar{A} = A/\tilde{C}$ ,  $\bar{B} = B/\tilde{C}$  and  $\bar{C} = C/\tilde{C}$ ;  $G$  decomposes as the extension:

$$1 \longrightarrow \tilde{C} \longrightarrow A *_C B \longrightarrow \bar{A} *_\bar{C} \bar{B} \longrightarrow 1$$

and with the Correspondence Theorem (Theorem 2.28, [Ro]), on the one hand  $[\bar{A} : \bar{C}] = [A : C]$  and  $[\bar{B} : \bar{C}] = [B : C]$  and on the other  $\bar{C}$  does not contain any non-trivial subgroup normal in  $\bar{A} *_\bar{C} \bar{B}$ . With the two key-points proved above,  $\bar{A} *_\bar{C} \bar{B}$  is icc. We conclude by applying the Theorem 4.1 after we have noted that an element  $x \in FC(\tilde{C})$  has a finite orbit under  $\pi(\bar{A} *_\bar{C} \bar{B})$  if and only if  $x$  has a finite orbit under  $\pi(A *_C B)$ .

By collecting all the above, we obtain:

**Proposition 6.2** (amalgams icc). *Let  $G = A *_C B$  be a non-trivial amalgamated product.*

• *In case the amalgam is non-degenerate:  $G$  is not icc if and only if:*

(i)  *$FC(\tilde{C})$  contains a non-trivial subgroup normal in  $\tilde{C}$  and  $\pi(G)$ -stable either finite or  $\mathbb{Z}^n$  and such that the image of  $\pi(G)$  in  $GL(n, \mathbb{Z})$  is finite.*

• *In case the amalgam is degenerate:  $G$  is not icc if and only if either (i) is satisfied (here  $\tilde{C} = C$ ) or the associated homomorphism  $\Theta : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Out}(C)$  is non-injective.*

*Example.* A free product of non-trivial groups is either icc or  $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ .

**6.3. Finite index subgroups.** Let  $H$  be a subgroup of the group  $G$  and  $\bar{H} := \bigcap_{g \in G} g H g^{-1}$ ;  $\bar{H}$  is indeed the maximal subgroup of  $H$  normal in  $G$ . If  $H$  has a finite index in  $G$ , let  $g_1, g_2, \dots, g_n$  be a finite set of representatives of  $G/H$ , then  $\bar{H} = \bigcap_{i=1}^n g_i H g_i^{-1}$ , and  $\bar{H}$  has a finite index in  $G$ . Denote by  $\pi$  the homomorphism  $\pi : G \rightarrow \text{Aut}(\bar{H})$  defined by  $\forall g \in G, k \in \bar{H}, \pi(g)(k) = gk$ .

**Proposition 6.3** (finite index subgroup and icc). *Let  $G$  be a group,  $H$  a finite index subgroup of  $G$  and  $\pi : G \longrightarrow \text{Aut}(\overline{H})$  the homomorphism defined above. Then:*

*$G$  is icc*

$\iff H$  is icc and  $\Theta : Q \longrightarrow \text{Out}(\overline{H})$  is injective,

$\iff H$  is icc and  $\forall g \in G \setminus H$  with finite order,  $\pi(g)$  is not the identity

$\iff H$  is icc and  $\forall g \in G \setminus H$  with finite order,  $\pi(g)$  is not inner.

*Example.* Virtually nilpotent groups are not icc. By a celebrated theorem of Gromov, finitely generated groups with polynomial growth are not icc.

*Proof.* First note that in an icc group every finite index subgroup is icc; for suppose that  $H$  is a finite index subgroup of  $G$  and that  $H$  is not icc: there exists  $h \in H \setminus \{1\}$  with  ${}^Hh$  finite, hence  $Z_H(h)$  has a finite index in  $H$ , and since  $H$  has a finite index in  $G$  and  $Z_G(h) \supset Z_H(h)$ ,  $Z_G(h)$  has a finite index in  $G$  and  $G$  is not icc. Hence  $G$  icc implies that  $H$  is icc.

Now suppose that  $H$  is icc, so that  $\overline{H}$ , which has finite index in  $H$ , is icc; apply proposition 5.8 to the extension:

$$1 \longrightarrow \overline{H} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

one obtains that  $G$  is icc if and only if  $\Theta : Q \longrightarrow \text{Out}(\overline{H})$  is injective. In particular if  $G$  is icc then  $\forall g \in G \setminus H$ ,  $\pi(g)$  is not inner, in particular is not the identity.

Now suppose that  $G$  is not icc, necessarily there exist  $g \in G \setminus \overline{H}$  and  $k \in K$  such that  $\forall x \in K$ ,  $\pi(g)(x) = {}^kx$ . Let  $\omega = gk^{-1}$ ,  $\pi(\omega)$  is the identity on  $\overline{H}$ ; note that  $\omega \in G \setminus H$  for  ${}^H\omega$  is finite and otherwise it would contradict that  $H$  is icc. Since  $\overline{H}$  has a finite index in  $G$ , there exists  $n > 1$  such that  $\omega^n \in \overline{H}$ . Since  $\overline{H}$  is icc, necessarily  $\omega^n = 1$ .

Finally let  $g \in G \setminus H$  such that  $g^n = 1$  and  $\exists k \in \overline{H}$  such that  $\forall x \in \overline{H}$ ,  $\pi(g)(x) = {}^kx$  and let  $u = gk^{-1}$ . Then  $\pi(u)$  is the identity on  $\overline{H}$  and  $u^n = g^n k^{-n} = k^{-n}$ ; necessarily  $u^n = 1$  for  $u^n$  lies in  $\overline{H}$  with a finite conjugacy class, hence  $u^n \neq 1$  would contradict the fact that  $H$  is icc.  $\square$

*Remark.* Under the same hypothesis  $G$  is icc if and only if  $H$  is icc and  $G \not\cong \overline{H} \times \mathbb{Z}/n\mathbb{Z}$ .

**Corollary 6.1.** *If  $G \setminus H$  contains no torsion element (in particular when  $G$  is torsion-free) then  $G$  is icc if and only if  $H$  is icc.*

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